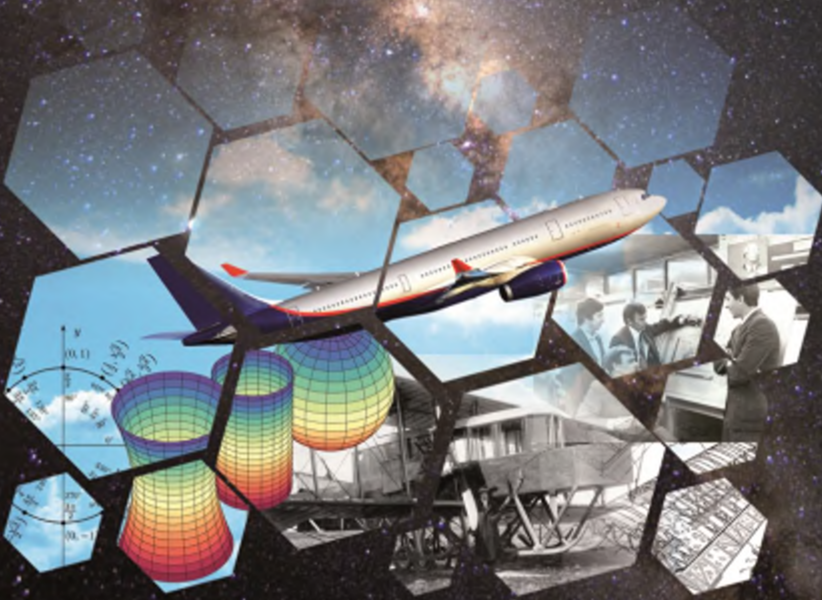
MOSCOW
AVIATION
INSTITUTE

NATIONAL
RESEARCH
UNIVERSITY

Textbook for Bachelor of Science Students

I. A. KUDRYAVTSEVA

CALCULUS OF A SINGLE VARIABLE



И. А. КУДРЯВЦЕВА

МАТЕМАТИЧЕСКИЙ АНАЛИЗ: ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЕ ФУНКЦИИ ОДНОЙ ПЕРЕМЕННОЙ

I. A. KUDRYAVTSEVA

CALCULUS OF A SINGLE VARIABLE

MOSCOW AVIATION INSTITUTE
(NATIONAL RESEARCH UNIVERSITY)

I. A. KUDRYAVTSEVA

CALCULUS OF A SINGLE VARIABLE

Moscow
«Dobroe Slovo»
2019

МОСКОВСКИЙ АВИАЦИОННЫЙ ИНСТИТУТ
(НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ)

И. А. КУДРЯВЦЕВА

МАТЕМАТИЧЕСКИЙ АНАЛИЗ: ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЕ ФУНКЦИИ ОДНОЙ ПЕРЕМЕННОЙ

Москва
Издательство «Доброе слово»
2019

УДК 512 (075.8)
ББК 22.143я73
К88

К88 Математический анализ: дифференциальное и интегральное исчисление функции одной переменной: Учебное пособие / И.А. Кудрявцева. – М.: Издательство «Доброе слово», 2019. – 160 с.: ил.

ISBN 978-5-89796-651-6

Пособие предназначено для ознакомительного, а также углубленного изучения курса математического анализа, посвещенному дифференциальному и интегральному исчислению функции одной переменной. Приведенный теоретический материал проилюстрирован большим количеством практических примеров. В конце каждого раздела предлагаются задачи для самостоятельного изучения.

Для студентов технических вузов и университетов.

K88 Calculus of a single variable: textbook / I.A. Kudryavtseva. – M.: «Dobroe Slovo», 2019. – 160 p.

Moscow Aviation Institute (National Research University). Moscow. E-mail: kaf805@mai.ru

Given textbook is written for supporting the first semester of calculus course and self-training students earning a bachelor degree in engineering. The materials embrace main topics of calculus of a single variable. Theoretical concepts presented in the book are illustrated by sufficient amount of examples and complemented by practical exercises.

For students of MAI International Bachelor's Degree Programs.

Ответственный редактор д-р физ.-мат. наук А.В. Пантелеев

CONTENTS

CHAPTER 1. INTRODUCTION TO CALCULUS

1.1. SETS AND SET OPERATIONS

Def.: A ***set*** is a collection of objects which can be defined so that it is definitely understood that any object is either in the set or not.

Examples of sets are:

1. A collection of numbers using for counting (the set of natural numbers).
2. The collection of solutions of the quadratic equation: $x^2-5x+6=0$.
3. A collection of functions that are continuous on $[a,b]$.

A set is usually denoted by capital letters A,B,C, while the objects, which compose the set, are denoted by small letters a,b,c.

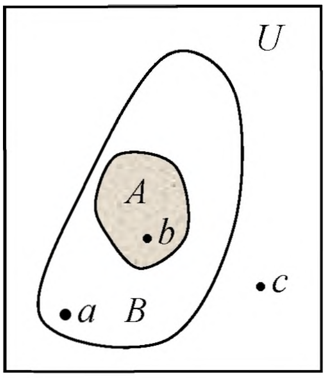
Def.: The objects are called ***members*** or ***elements*** of a set.

Def.: The set with no elements is called an ***empty*** set and is denoted by $\varnothing$.

Def.: A set, which contains all the elements of other given sets, is called a ***universal set***. The symbol for denoting a universal set is U.

Graphical representation of sets

Mathematician John Venn has introduced the concept of graphical set representation by means of closed geometrical figures, which are called ***Venn diagrams***. Venn diagrams are useful for solving simple logical problems. In Venn diagrams, the universal set U is represented by a rectangle and all other sets are represented by circles within the rectangle.



Pic. 1.1

Pic. 1.1 shows two sets A,B and three points a, b, c. The point a is an element of the set B, b is an element of sets A and B at the same time, point c does not belong neither to A nor to B.

It can be denoted as follows:

$$a \notin A \qquad b \in A,\ b \in B \qquad c \notin A \quad c \notin B.$$

Set specification

There are two ways of set ***specification***.

• One way consists in ***listing*** elements of a set. The correct representation of a set is to write the elements, separated by commas and enclosed between braces or curly brackets.

Example 1.1.

1. The set of natural numbers $\mathbb{N}$ can be defined by listing its elements: $\mathbb{N}=\{1,2,3,...\}$.

2. A number set whose elements are values of terms of the arithmetic progression with the initial term $a_1 = 1$ and the common difference $d = \frac{1}{2}$. Since the nth term of an arithmetic progression $a_n = a_{n-1} + d = a_1 + (n-1)d$, the desired number set $A = \left\{1, \frac{3}{2}, 2, \ldots\right\}$.

● The second way of set specification consists in the ***definition of the rule or property***, which characterizes the set.

Example 1.2. Specify the collection of the quadratic equation solutions $x^2 - 5x + 6 = 0$ using both ways.

□ Let A denote a set of the solutions of the equation $x^2 - 5x + 6 = 0$. Then $x_1 = 2$ and $x_2 = 3$ are the desired roots.

Hence, according to the first way of specification $A = \{2;3\}$.

The second way gives us $A = \{x \mid x^2 - 5x + 6 = 0\}$ or $A = \{x : x^2 - 5x + 6 = 0\}$.■

Remark 1.1

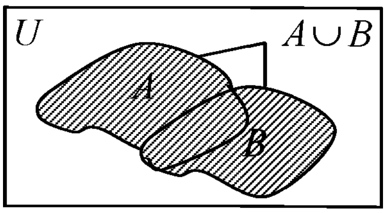
Note, the stroke | or colon : can be used interchangeably; they mean ‘such that’. The representation $A = \{x \mid x^2 - 5x + 6 = 0\}$ is read as follows: A is a set of such elements x, that $x^2 - 5x + 6 = 0$.

Def.: A set A is said to be ***a subset*** of a set B and denoted by $A \subseteq B$ if every element of A is an element of B.

Def.: Sets A and B are said to be ***equal*** if and only if $A \subseteq B$ and $B \subseteq A$.

Set operations

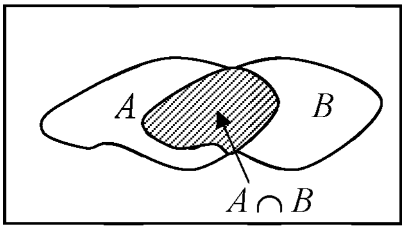
Def.: The ***union*** of two sets A and B ($A \cup B$) is a set of elements that belong to A or B: $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Below (pic. 1.2) the result of the union operation is illustrated by use of Venn diagrams.



Pic. 1.2

Def.: The ***intersection*** of two A and B ($A \cap B$) is a set of elements that belong to both A and B: $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

In pic. 1.3 the result of the intersection operation is given.



Pic. 1.3

Example 1.3. Let $A = \{1;2;3;4;5\}$ and $B = \{2;4;6;8;10\}$. Find $A \cup B$ and $A \cap B$.

□ $A \cup B = \{1; 2; 3; 4; 5; 6; 8;10\}$; $A \cap B = \{2; 4\}$. ■

Def.: The ***relative compliment*** of A in B ($B \setminus A$ or $B - A$) is a set of all elements that don't belong to A but belong to B: $B \setminus A = \{x: x \in B \text{ and } x \notin A\}$ (pic. 1.4,*a*).

Def.: The ***absolute compliment*** of A in U ($\overline{A}$) is a set of all elements that don't belong to A: $\overline{A} = \{x: x \notin A\}$ (pic. 1.4, *b*).

Pic. 1.4

Properties of set operations

1) $A \cup B = B \cup A$ (Commutative law),
 $A \cap B = B \cap A$,
2) $A \cup \varnothing = A$ (Identity law),
 $A \cap U = A$,
3) $A \cup A = A$ (Idempotent law),
 $A \cap A = A$,
4) $A \cup U = U$ (Domination law)
 $A \cap \varnothing = \varnothing$
5) $A \cup (B \cup C) = (A \cup B) \cup C$, (Associative law),
 $A \cap (B \cap C) = (A \cap B) \cap C$,
6) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (Distributive law),
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
7) $\overline{\overline{A}} = A$, (Involution law),
8) $\overline{A \cap B} = \overline{A} \cup \overline{B}$, (De Morgan's law),
 $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Def.: Sets A and B are ***disjoint*** sets, if they do not have common elements: $A \cap B = \varnothing$.

Remark 1.2

If A and B are disjoint sets, then $A \setminus B = A$ and $B \setminus A = B$.

Example 1.4. Let $A = \{1;2;3;4;5\}$ and $B = \{2;4;6;8;10\}$. Find $A \setminus B$ and $B \setminus A$.

□ According to the given definitions: $A \setminus B = \{1;3;5\}$, $B \setminus A = \{6;8;10\}$. ■

Example 1.5. Sets A, B, C can be described as:

$A = \{x: x$ is a natural number between 1 and 5$\}$,

$B = \{x: x$ is an even number between 1 and 5$\}$,

$C = \{x: x$ is an odd number between 1 and 5$\}$.

Find $A \cup B$, $B \cup C$, $A \cup B \cup C$, $A \cap B$, $B \cap C$, $A \cap C$, $A \setminus B$, $B \setminus C$, $C \setminus A$.

□ According to the given descriptions:

$$A = \{1;\ 2;\ 3;\ 4;\ 5\},\ B = \{2;\ 4\},\ C = \{1;\ 3;\ 5\}$$

Hence, unions of sets:

$$A \cup B = \{1;\ 2;\ 3;\ 4;\ 5\} = A, \quad B \cup C = \{1;\ 2;\ 3;\ 4;\ 5\} = A,$$

$$A \cup B \cup C = \{1;\ 2;\ 3;\ 4;\ 5\} = A.$$

Intersections of sets:

$$A \cap B = \{2;\ 4\} = B, \quad B \cap C = \varnothing, \quad A \cap C = \{1;\ 3;\ 5\} = C.$$

Compliments:

$$A \setminus B = \{1;\ 3;\ 5\} = C, \quad B \setminus C = \{2;\ 4\} = B, \quad C \setminus A = \varnothing. \ ■$$

1.2. NUMBER SETS. COMPLEX NUMBERS

Number sets

The most important sets, which are considered in mathematics, are sets of numbers or ***number sets***:

1. The set of natural numbers (or natural numbers) $\mathbb{N}$: $\mathbb{N} = \{1;2;...;n,...\}$.
2. The set of integer numbers (or integers) $\mathbb{Z}$: $\mathbb{Z} = \{...,-1;\ 0;\ 1;\ 2;...;n,...\}$.
3. The set of rational numbers (or rational numbers) $\mathbb{Q}$:

$$\mathbb{Q} = \left\{\frac{m}{n}: m \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

4. The set of irrational numbers (or irrational numbers) I. The set I contains numbers that can't be expressed as a fraction $\frac{m}{n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and their decimal form involves an infinite sequence of numerals without repeating patterns.

For example, $\sqrt{2} = 1.41421356237309504880168$. There is no repeating in decimal places in comparison with, for example, the number $\frac{1}{3} = 0.33333333333333333333333$.

5. The set of real numbers (or real numbers) $\mathbb{R}$: $\mathbb{R} = \mathbb{Q} \cup \mathrm{I}$.

Some characteristics of number sets

Def.: A set $A \subseteq \mathbb{R}$ is said to be ***bounded from above*** if there exists a number $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in A$, i.e.,

A is bounded from above $\Leftrightarrow \exists M \in \mathbb{R} : \forall a \in A \; a \leq M$.

The number M is called an ***upper bound*** of A.

Def.: A set A is said to be ***bounded from below*** if there exists a number m such that $a \geq m$ for all $a \in A$, i.e.,

A is bounded from below $\Leftrightarrow \exists\, m \in \mathbb{R} : \forall a \in A \; a \geq m$.

The number m is called a ***lower bound*** of A.

Def.: A set A is said to ***be bounded*** if it is bounded both from above and below.

Note that the upper bound and the lower bound are not unique. Apparently, if M is an upper bound, then values $M+1, M+2,$ and so on are also upper bounds.

Proposition 1.1.

A set A is bounded if and only if $\exists M \in \mathbb{R}: \quad |a| \leq M \qquad \forall a \in A.$

Complex numbers

Suppose, it is needed to find roots of the equation: $x^2 + 1 = 0$. Apparently it has no real roots. However, to solve the problem we can introduce a new set $\mathbb{C}$ that may be considered as an expansion of the real number set $\mathbb{R}$. For this reason a new element $i \in \mathbb{C}$ such that $i^2 = -1$ should be introduced. The element i is called ***the imaginary unit***. Thus the given equation has two roots: $i; -i$.

Def.: A number z expressed in the form $x + iy$ where $x, y \in \mathbb{R}$, i is the imaginary unit is called a ***complex number***. The form $x + iy$ is referred to as the ***algebraic form*** of the complex number z.

Def.: x is called the ***real part*** of z: $x = \operatorname{Re} z$. y is called the ***imaginary part*** of z: $y = \operatorname{Im} z$.

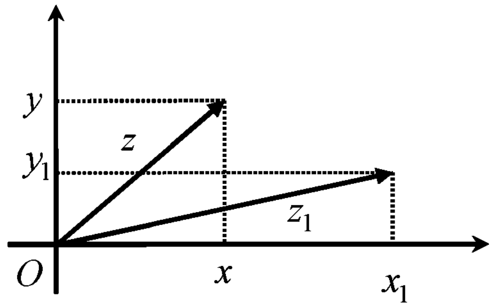
Example 1.6. Find Rez, Imz, if a) $z = 1 + i$; b) $z = 4$;c) $z = -\frac{i}{2}$.

□ a) If $z = 1 + i$ then $\operatorname{Re} z = 1$, $\operatorname{Im} z = 1$.

b) If $z = 4$ then $\operatorname{Re} z = 4$, $\operatorname{Im} z = 0$.

c) If $z = -\frac{i}{2}$ then $\operatorname{Re} z = 0$, $\operatorname{Im} z = -\frac{1}{2}$.■

So to determine a complex number z an ordered pair of real numbers (x, y) should be taken. Furthermore as we know a geometric image of a real number is a point on the real line. Then a geometric image of an ordered pair (x, y) is a point or its radius vector on the coordinate plane (pic. 1.5).



Pic. 1.5

Def.: The coordinate plane where complex numbers are depicted is called the ***complex plane***. The x-axis is called the ***real line***, the y-axis is called the ***imaginary line***.

Operations on complex numbers expressed in the algebraic form

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers. Then

1. $z_1 = z_2$ if and only if $x_1 = x_2$, $y_1 = y_2$.

It should be remembered that nevertheless examining two complex numbers for equality can be possible comparison operation is inapplicable.

2. $z = z_1 \pm z_2$ if $z = x + iy$, where $x = x_1 \pm x_2, y = y_1 \pm y_2$.

3. $z = z_1 \cdot z_2$ if $z = x + iy$, where $x = x_1 \cdot x_2 - y_1 \cdot y_2, y = x_1 y_2 + x_2 y_1$.

4. $z = \frac{z_1}{z_2}$ if $z = x + iy$, where $x = \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{x_2^2 + y_2^2}, y = \frac{y_1 \cdot x_2 - x_1 \cdot y_2}{x_2^2 + y_2^2}$.

Def.: A complex number $\bar{z} = x - iy$ is called a ***complex conjugate*** of $z = x + iy$.

Example 1.7. Find $z_1 + z_2, z_1 - z_2, z_1 \cdot z_2, \frac{z_1}{z_2}$, if $z_1 = 1 + i$, $z_2 = 1 - 2i$.

□ Operating with complex numbers can be carried out as with algebraic expressions. $z_1 + z_2 = 1 + i + 1 - 2i = (1+1) + i(1-2) = 2 - i$. In like manner we have $z_1 - z_2 = 1 + i - (1 - 2i) = 3i$.

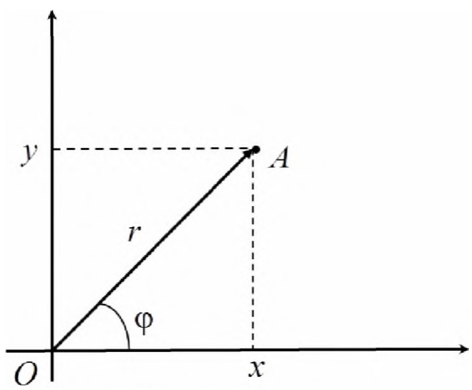
Since $i^2 = -1$, $z_1 \cdot z_2 = (1+i)(1-2i) = = 1 - 2i^2 - 2i + i = 3 - i$.

To find the quotient $\frac{z_1}{z_2}$ the formula $\frac{z_1}{z_2}=\frac{z_1\overline{z}_2}{z_2\overline{z}_2}$ can be used. Note, that for any complex number $z=x-iy$ the product $z\overline{z}$ results in x^2+y^2. Then, $\frac{z_1}{z_2}=\frac{z_1\overline{z}_2}{z_2\overline{z}_2}=\frac{(1+i)(1+2i)}{1+4}=\frac{1-2+2i+i}{5}=-\frac{1}{5}+\frac{3}{5}i$.■

Example 1.8. Find $(1+i)^4$.

□ Using the Binomial formula in the form $(x+iy)^n=\sum_{i=0}^{n}C_n^i x^{n-i}(iy)^i$, where the binomial coefficients $C_n^i=\frac{n!}{i!(n-i)!}$, $n!=1\cdot 2\cdot\ldots\cdot n$, we have $(1+i)^4=$ $=C_4^0+C_4^1 i+C_4^2 i^2+C_4^3 i^3+C_4^4 i^4$. Since $C_4^0=\frac{4!}{0!4!}=1$, $C_4^1=\frac{4!}{1!3!}=4$, $C_4^2=\frac{4!}{2!2!}=6$, $C_4^3=\frac{4!}{3!1!}=4$, $C_4^4=\frac{4!}{4!0!}=1$ and $i^2=-1$, $i^3=i\cdot i^2=-i$, $i^4=1$, we get $(1+i)^4=1+4i-6-4i+1=-4$.■

A position of a point on the coordinate plane (pic. 1.6) can be determined not only by an ordered pair (x,y) but the ordered pair (r,φ), where r is the distance from the origin to the point (OA) and φ is the angle between OA and the positive direction of the x-axis.



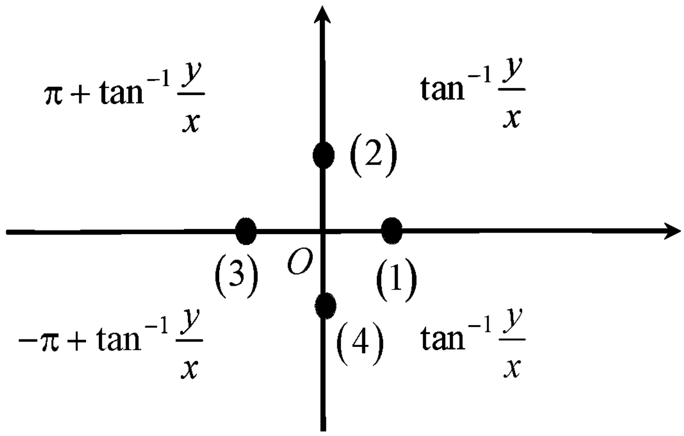
Pic. 1.6

Def.: r is denoted by $|z|$ and called the ***modules*** of a complex number z.

As shown in pic.1.6, r or OA is a hypotenuse of ΔAOB then r or $|z|$ is $\sqrt{x^2+y^2}$. Angle φ can be found from the equation $\tan\varphi=\frac{y}{x}$.

However, the correspondence between points on the plane and pairs (r,φ) is not a one-to-one correspondence. For example, the pairs $\left(1,\frac{\pi}{4}\right)$ and $\left(1,\frac{9\pi}{4}\right)$ define a position of the same point.

To disambiguate this fact the range of φ should be restricted. Let φ vary from $-\pi$ to π. To find φ it is convenient to use the scheme (pic. 1.7):



$$(1): \varphi = 0; \ (2): \varphi = \frac{\pi}{2}; \ (3): \varphi = \pi; \ (4): \varphi = -\frac{\pi}{2}$$

Pic. 1.7

Def.: φ is called the ***principle value of the argument*** of z and denoted by $\arg z$.

All possible values $\operatorname{Arg} z$ of the argument of z are given by

$$\operatorname{Arg} z = \arg z + 2\pi k, \ k \in Z.$$

Consider the problem of evaluating x and y in terms of r and φ. Concerning the fact that the side OB being adjacent to $\angle AOB$ is x and the hypotenuse OA is r we have $x = OA\cos(\angle AOB) = r\cos\varphi$. By analogy, $y = OA\sin(\angle AOB) = r\sin\varphi$

Then, any complex number $z = x + iy$ can be represented in the form

$$z = x + iy = r(\cos\varphi + i\sin\varphi).$$

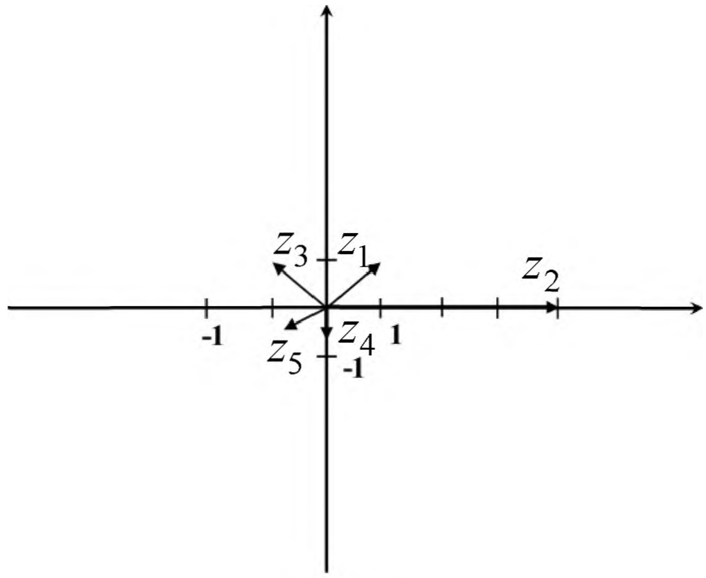
This form is called the ***trigonometric form*** of a complex number.

Example 1.9. Find modules and arguments of the given complex numbers. Plot geometric images of the numbers, if

$$z_1 = 1 + i, \ z_2 = 4, \ z_3 = -1 + i, \ z_4 = -\frac{i}{2}, \ z_5 = -\frac{\sqrt{3}}{2} - \frac{i}{2}.$$

□ By the definition of the modulus, $|z| = \sqrt{x^2 + y^2}$, where $x = \operatorname{Re} z, y = \operatorname{Im} z$. Then for $z_1 = 1 + i$ we get $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Using the scheme presented on pic. 1.7 $\varphi_1 = \tan^{-1}\frac{y}{x} = \arctan\frac{y}{x} = \arctan 1 = \frac{\pi}{4}$. In like manner, $|z_2| = \sqrt{4^2 + 0^2} = 4$, $\varphi_2 = 0$; $|z_3| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\varphi_3 = \pi + \arctan\frac{1}{-1} = \frac{3\pi}{4}$; $|z_4| = \sqrt{0^2 + \left(-\frac{1}{2}\right)^2} = \frac{1}{2}$, $\varphi_4 = -\frac{\pi}{2}$;

$|z_5|=\sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=1$, $\varphi_1=-\pi+\arctan\frac{1}{\sqrt{3}}=-\frac{5\pi}{6}$. The images are depicted in pic. 1.8.■



Pic. 1.8

Operations on complex numbers expressed in the trigonometric form

Let $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$ and $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$ be two complex numbers represented in the trigonometric form.

Then

1. $z=z_1\cdot z_2=r_1(\cos\varphi_1+i\sin\varphi_1)\cdot r_2(\cos\varphi_2+i\sin\varphi_2)=$
$=r_1r_2\{(\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2+\sin\varphi_2\cos\varphi_1)\}=$
$=r_1r_2\{\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)\}$;

2. $z=\frac{z_1}{z_2}=\frac{r_1}{r_2}\{\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)\}$;

3. $z=z_1^n=r_1^n\{\cos(n\cdot\varphi_1)+i\sin(n\cdot\varphi_1)\}$, $n\in N$.

In the particular case, when $r_1=1$, we have

$$\{\cos\varphi_1+i\sin\varphi_1\}^n=\{\cos(n\cdot\varphi_1)+i\sin(n\cdot\varphi_1)\}.$$

This formula is called by de ***Moivre's formula***.

4. $z=\sqrt[n]{z_1}=\sqrt[n]{r_1}\left\{\cos\left(\frac{\varphi_1+2\pi k}{n}\right)+i\sin\left(\frac{\varphi_1+2\pi k}{n}\right)\right\}$, $k=0,\ldots,n-1$, $n\in\mathrm{N}$.

Example 1.10. Find $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5$.

□ To raise the complex number $2+i$ to 5 it is needed to represent it in the trigonometric form: $\frac{\sqrt{3}}{2}+\frac{i}{2}=\sqrt{1}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, wherefrom $r=1, \varphi=\frac{\pi}{6}$. Then $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=1^5\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}+\frac{i}{2}$.■

Example 1.11. Find $(1+i)^5\left(\sqrt{3}-i\right)^7$.

□ The given expression can be considered as a product of two numbers $z_1=(1+i)^5$ and $z_2=\left(\sqrt{3}-i\right)^7$. According to example 1.9,

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right);\ \sqrt{3}-i=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right).$$

Then

$$z_1=(1+i)^5=4\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right); z_2=\left(\sqrt{3}-i\right)^7=$$

$$=128\left(\cos\left(-\frac{7\pi}{6}\right)+i\sin\left(-\frac{7\pi}{6}\right)\right),$$

wherefrom $r_1=4\sqrt{2},\ r_2=128$. However, it should be noticed that $\frac{5\pi}{4}$ and $-\frac{7\pi}{6}$ can not be taken for the arguments φ_1 and φ_2, because the argument of a complex number φ varies from $-\pi$ to π. Thus, reduction formulas should be used: $\frac{5\pi}{4}=2\pi-\frac{3\pi}{4}$ and $-\frac{7\pi}{6}=-2\pi+\frac{5\pi}{6}$. Then, $\varphi_1=-\frac{3\pi}{4}$ and $\varphi_2=\frac{5\pi}{6}$.

In result, $(1+i)^5\left(\sqrt{3}-i\right)^7=4\sqrt{2}\cdot 128\left(\cos\left(\frac{5\pi}{6}-\frac{3\pi}{4}\right)+i\sin\left(\frac{5\pi}{6}-\frac{3\pi}{4}\right)\right)=$

$=512\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$.■

Example 1.12. Solve the equation $z^3=1+i$

□ To find all roots of the given equation it is needed to calculate all values of $\sqrt[3]{1+i}$. For this reason the complex number $1+i$ should be represented in the trigonometric form: $1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$, wherefrom $r=\sqrt{2}, \varphi=\frac{\pi}{4}$. Then,

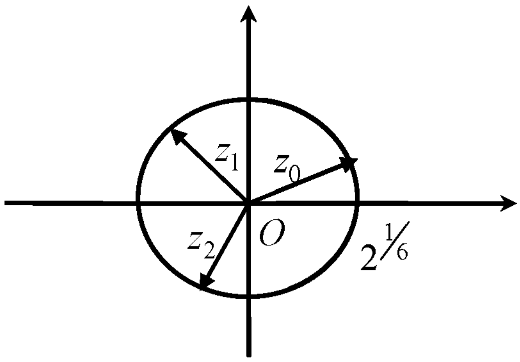
$$\sqrt[3]{1+i}=\sqrt[3]{\left(\sqrt{2}\right)}\left\{\cos\left(\frac{\frac{\pi}{4}+2\pi k}{3}\right)+i\sin\left(\frac{\frac{\pi}{4}+2\pi k}{3}\right)\right\},\ k=0,1,2,\ \text{i.e.}$$

$$k=0,\ z_0=2^{\frac{1}{6}}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)\approx 1.084+0.291i,$$

$$k=1,\ z_1=2^{\frac{1}{6}}\left(\cos\frac{9\pi}{12}+i\sin\frac{9\pi}{12}\right)\approx -0.794+0.794i,$$

$$k=2,\ z_2=2^{\frac{1}{6}}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)\approx -0.291-1.084i.$$

The obtained solutions are plotted in pic. 1.9.■



Pic. 1.9

Exercises

1. Find $\operatorname{Im}\overline{z}$, if $z=\dfrac{i}{1-2i}$.
2. Find $\operatorname{Re}\overline{z}$, if $z=\left(\dfrac{2-i}{1+i}\right)^3$.
3. Find $|z_1\cdot z_2|$, $\arg z_2$, $\operatorname{Re}(z_1\cdot z_2)$, $\operatorname{Im}\left(\dfrac{z_1}{z_2}\right)$, if $z_1=1+i^{123}$, $z_2=-2+2i$.
4. Find modulus and arguments of the complex numbers:
   $z_1=-2i\cdot\left(\cos\frac{4\pi}{7}-i\sin\frac{4\pi}{7}\right)$; $z=(1+i)\left(\sqrt{3}-i\right)$.
5. Solve the equations: $z^6-1=0$; $z^3-i=0$.
6. Depict the regions on the complex plane given by

   a) $|z|=1$; b) $\begin{cases}|z|=1,\\ 0\le\arg z\le\frac{\pi}{2};\end{cases}$ c) $\begin{cases}\arg z=\frac{\pi}{4},\\ |z|\le 1.\end{cases}$

1.3. CONCEPT OF A FUNCTION

Def.: A function f from a set X to a set Y is a correspondence that assigns to each element x of X a unique element y of Y. The element x is called an ***independent variable*** or an ***argument.*** The element y is called a ***depended variable*** or an ***image*** of x under f and denoted by $f(x)$.

Def.: The set X is called the ***domain*** of the function f. The ***range*** of the function f consists of all images of elements of X.

Remark 1.3

1. It is generally said that a function $y = f(x)$ ***maps*** a set X into a set Y: $f{:}X \to Y$. In this case f is called a ***mapping*** of X into Y.

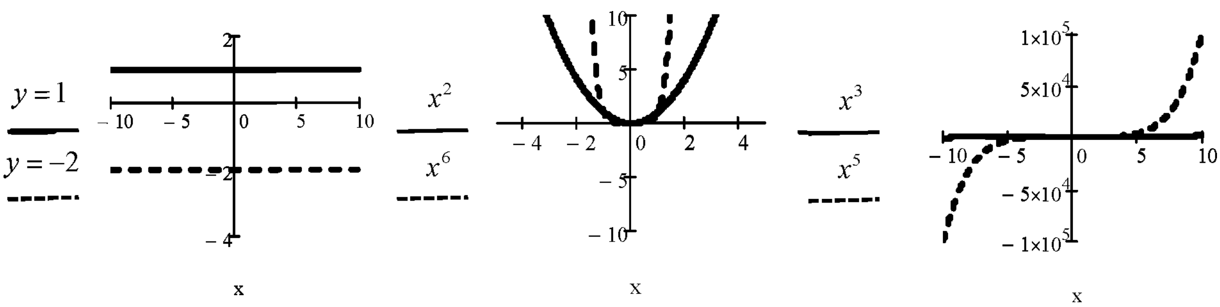
2. The symbol $f(x)$ is used for the element associated with x, and it is read "f of x". Sometimes $f(x)$ is called the ***value*** of f at x.

3. In particular, we may use any other letter instead of x as an argument of f. For example, the functional correspondences: $f(x) = x^2$, $f(t) = t^2$ and $f(\alpha) = \alpha^2$ are identical and defined the mapping $f: \mathbb{R} \to \mathbb{R}_+$, where $\mathbb{R}_+$ is a set of all nonnegative real numbers. Also, we may use any letter instead of f for function denotation. The functions $g(x) = x^2$, $u(t) = t^2$ and $f(x) = x^2$ are also identical.

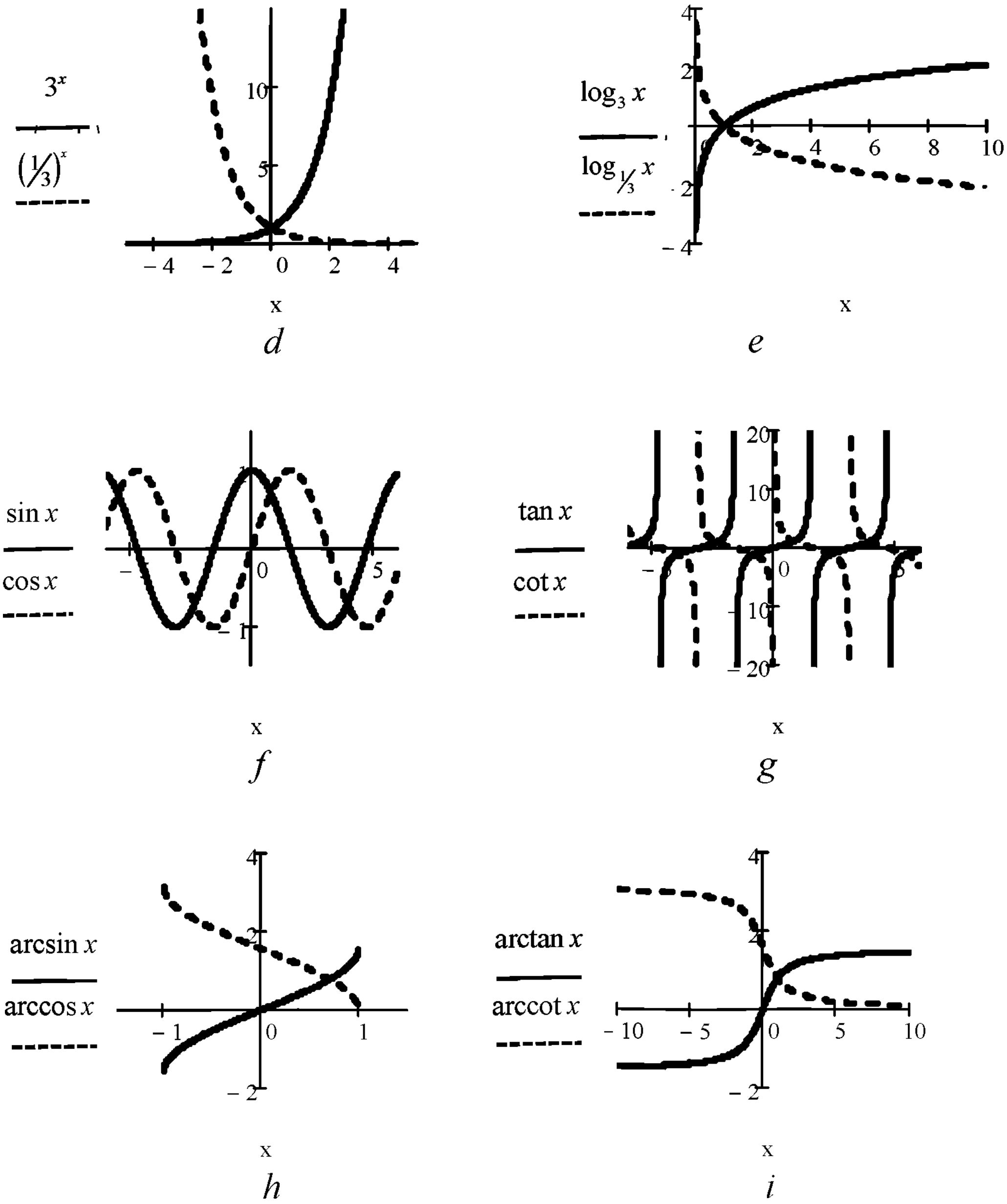
Classification of functions

Functions listed below are called basic elementary functions:

- constant C (pic.1.10, *a*),
- the power function x^α (pic.1.10, *b–c*),
- the exponential a^x (pic.1.10, *d*),
- the logarithm $\log_a x$ (pic.1.10, *e*),
- the trigonometric functions: $\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$ (pic.1.10,*f–g*),
- the inverse trigonometric functions: $\arcsin(x)$ ($\sin^{-1}(x)$), $\arccos(x)$ ($\cos^{-1}(x)$), $\arctan(x)$ ($\tan^{-1}(x)$), $\text{arccot}(x)$ ($\cot^{-1}(x)$) (pic.1.10, *h–i*).



a *b* *c*



Pic. 1.10

Def.: ***An elementary function*** is a function that may be represented by a single formula $y = f(x)$, where $f(x)$ involves only a finite number of arithmetic operations (addition, subtraction, multiplication, division) on basic elementary functions and expressions that are functions of functions called ***composite functions***.

Examples of elementary functions:

- the hyperbolic functions:

the hyperbolic sine $\operatorname{sh}x = \dfrac{e^x - e^{-x}}{2}$,

the hyperbolic cosine $\operatorname{ch}x = \dfrac{e^x + e^{-x}}{2}$, the hyperbolic tangent $\text{th}x = \dfrac{\text{sh}x}{\text{ch}x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$,

the hyperbolic cotangent $\text{cth}x = \dfrac{\text{ch}x}{\text{sh}x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$;

- the rational functions:

  the linear function $y = ax + b$,

  the quadratic function $y = ax^2 + bx + c$,

  the polynomial functions

$$P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n, \quad Q_m(x) = b_0 x^m + b_1 x^{m-1} + \ldots + b_m,$$

the rational function $R(x) = \dfrac{P_n(x)}{Q_m(x)}$.

Other examples of elementary functions:

$$y = \sin\left(x^2\right), \quad y = \sqrt[5]{x^2 + 3x^4}, \quad y = (x-1)e^x, \; y = (1+x)^{\frac{1}{x}}.$$

Examples of ***non-elementary functions***:

- $1 + x + x^2 + \ldots + x^{n-1} + \ldots$

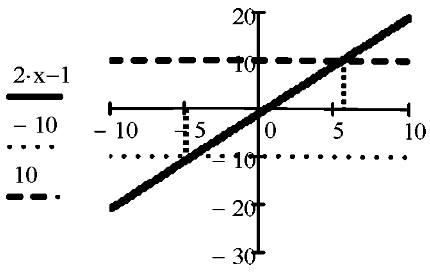
This formula contains an infinite number of arithmetic operations;

- $y = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$

Inspite of the fact that the latter function called a ***piecewise function*** coincides with basic elementary functions in separate parts of the domain, it is not an elementary function in the entire domain.

Inverse functions

Consider the function $y = f(x)$, where $f(x) = 2x - 1$. The graph of f is illustrated below (pic. 1.11).



Pic. 1.11

As we see the considered function satisfies the condition: $f(x_1)=f(x_2)$ implies $x_1=x_2$. Graphically it means that any horizontal line $y=C$ intersects the graph no more that at one point. Consequently the equation $y=f(x)$ has no more than one solution for each y. Solving $y=f(x)$ for x we get the correspondence $x=g(y)$ that is called an ***inverse function***. Also it can be possible another denotation for the inverse function: $x=f^{-1}(y)$.

Composite functions (functions of a function)

Def.: Consider a function $u=\varphi(x)$, which maps a set X onto a set U: $\varphi{:}X\to U$, and a function $y=f(u)$, which maps a set U onto a set Y: $f{:}U\to Y$, then the ***composite function*** $y=f(\varphi(x))$ maps a set X onto a set Y: $f\circ\varphi: X\to Y$.

Remark 1.4

We read function notation $(f\circ\varphi)(x)$ from right to left that means we should calculate the value of φ at x first then substitute the result in f.

Example 1.13. Find composite functions $f\circ\varphi$ and $\varphi\circ f$, if $f(x)=3x^2-4$ and $\varphi(x)=\sqrt{x-1}$.

□ According to the definition $(f\circ\varphi)(x)=f(\varphi(x))$ while $(\varphi\circ f)(x)=\varphi(f(x))$.Thus,

$$f(\varphi(x))=f\left(\sqrt{x-1}\right)=3\left(\sqrt{x-1}\right)^2-4=3(x-1)-4=3x-7\text{, if } x-1\ge 0$$

$$\varphi(f(x))=\varphi\left(3x^2-4\right)=\sqrt{\left(3x^2-4\right)-1}=\sqrt{3x^2-5}\text{, if } 3x^2-5\ge 0\text{ . ■}$$

Example 1.14. Represent the function $y=\sqrt{x-1}$ as a composition of elementary functions.

□ Let $\varphi(x)=x-1$, $f(u)=\sqrt{u}$. Notice, $\varphi(x)$ and $f(u)$ are elementary functions. Then $y=\sqrt{x-1}=f(\varphi(x))$ or $y=(f\circ\varphi)(x)$.■

The ways of functions representation

The following ways can be classified:

1) analytical representation,
2) table representation,
3) graphical representation,
4) representation by verbal description.

Analytical representation. The functions are represented analytically by means of formulas:

- $y = f(x)$ – this equation specifies an ***explicit function***,
- $F(x,y) = 0$ – this equation specifies an ***implicit function***,
- $x = \varphi(t),\ y = \psi(t)$ – these equations specify a ***parametrically*** defined function.

Table representation of a function. Let $\{x_1, x_2, ..., x_n\}$ be a set of ordered values of arguments, where $x_1 < x_2 < ... < x_n$, $\{y_1, y_2, ..., y_n\}$ – a set of corresponding values of a function. The function can be represented by the ***table*** shown below:

Table 1.1

x	x_1	x_2	x_3	x_4	...	x_n
y	y_1	y_2	y_3	y_4	...	y_n

Graphical representation of a function. Consider a function $f(x)$ defined as $f\colon X \to Y \Leftrightarrow f(x) = y$.

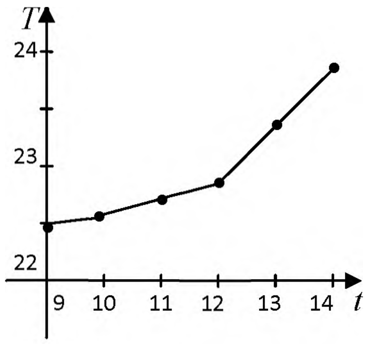
Def.: The ***graph*** of the function $f(x)$ is called a set of ordered pairs: $G = \{(x,y) \in \mathbb{R}^2 : x \in X,\ y = f(x)\}$. A point on the xy-plane is assigned to an ordered pair $(x,y) \in G$.

Verbal description. Define a function $f(x)$ as the integer part of a number x: $f(x) = [x]$. The largest integer that does not exceed x is called the integer part of the number x (denoted by $[x]$). Thus, [1,2] = 1, [2] = 2, [–2,3] = –3. The function $f(x) = [x]$ is called the ***floor function***. The domain is the set of real numbers $\mathbb{R}$, the range is the set of integer numbers $\mathbb{Z}$.

Example 1.15 . The dependence between the temperature T and the time t is represented below by means of the table 1.2 and the graph (pic. 1.12).

Table 1.2

t	9:00	10:00	11:00	12:00	13:00	14:00
T	21°	21,2°	21,5°	21,8°	22,8°	23,5°



Pic.1.12

Graphing functions

To plot the graph it is useful to specify some ***properties of a function***.

Def.: The domain $D(f)$ of a function $f(x)$ is called a ***symmetric*** set if for every $x \in D(f)$ there exists $-x$ such that $-x \in D(f)$.

Def.: The function $f(x)$ is called ***even*** if $f(-x) = f(x)$ for every $x \in D(f)$. The function $f(x)$ is called ***odd*** if $f(-x) = -f(x)$ for every $x \in D(f)$. The graph of an even function is symmetric about the y-axis. The graph of an odd function is symmetric with respect to the point O (the origin).

Def.: A ***periodic*** function is a function that repeats its values in regular intervals or periods. A function f is said to be periodic with period T $(T > 0)$ if $f(x-T) = f(x) = f(x+T)$ for any $x \in D(f)$. The smallest positive constant T (if it exists) is called a ***basic period***.

Example 1.16 . Sketch the graph of the function $f(x) = \sqrt{1-x}$.

□ The function $f(x) = \sqrt{1-x}$ is an ***explicit*** function. The domain $D(f)$ can be defined by the inequality $1 - x \geq 0$, i.e. $D(f) = \{x: x \leq 1\} = (-\infty, 1]$. Notice, that the domain $D(f)$ is not a symmetric set. Consequently, the function f is neither an even nor an odd function. Also $f(x) = \sqrt{1-x}$ is not a periodic function.

To sketch the graph we can use the table of graph transformations. Suppose, the form of the graph of $f(x)$ is known then to sketch the graph $f(x+C)$, where $C > 0$, we should shift the graph of $f(x)$ C units to the left.

There exist other ***transformations*** listed below (table 1.3).

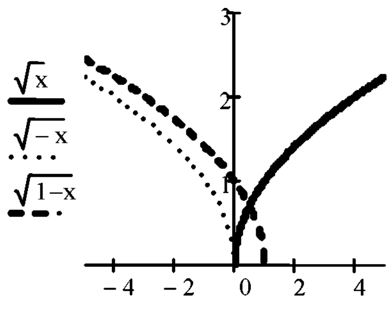
Table 1.3

No	Function	Transformation
1	$f(x+C)$	Shift the graph of $f(x)$ C units to the left if $C>0$. Shift the graph of $f(x)$ C units to the right if $C<0$.
2	$f(x)+C$	Shift the graph of $f(x)$ C units up if $C>0$. Shift the graph of $f(x)$ C units down if $C<0$.
3	$f(Cx)$	Stretch the graph of $f(x)$ horizontally by C, if $C<1$. Shrink the graph of $f(x)$ horizontally by C, if $C>1$.
4	$Cf(x)$	Stretch the graph of $f(x)$ vertically by C, if $C>1$. Shrink the graph of $f(x)$ vertically by C, if $C<1$.
5	$-f(x)$	Reflect the graph of $f(x)$ over the x-axis
6	$f(-x)$	Reflect the graph of $f(x)$ over the y-axis

Function $\sqrt{x}$ can be taken as the original function whose graph is well known. Write out the sequence of transformations:

$$\sqrt{x}\overset{6}{\Rightarrow}\sqrt{-x}\overset{1}{\Rightarrow}\sqrt{1-x}.$$

Above arrows number of transformation is marked. Steps of sketching the graph of the given function are shown below (pic. 1.13). ■



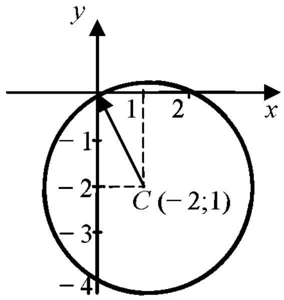
Pic. 1.13

Example 1.17. Prove that the equation $x^2+y^2=2x-4y$ determines a circle. Find its radius and coordinates of the center. Sketch the circle.

□ A ***circle*** can be defined as the locus of all points that satisfy the equation: $(x-x_0)^2+(y-y_0)^2=R^2$, where R $(R\ge 0)$ is the radius of the circle, and x_0, y_0 are the coordinates of its center. Rearranging terms and completing the square in the equation $x^2+y^2=2x-4y$ we have

$$x^2+y^2-2x+4y=0\Rightarrow\left(x^2-2x+1\right)+\left(y^2+4y+4\right)-5=0\Rightarrow(x-1)^2+(y+2)^2=5.$$

Hence, $x_0 = 1$, $y_0 = -2$, $R = \sqrt{5}$. Notice, that this equation specifies an implicit function. ■



Pic. 1.14

Graphs of functions represented parametrically

In some cases it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable, which is called a parameter: $x = \varphi(t)$, $y = \psi(t)$. In this case, any value of t generates a pair of values x and y, which is considered as a point of the curve.

Table 1.4

t	t_0	t_1	...
x	$x(t_0)$	$x(t_1)$	...
y	$y(t_0)$	$y(t_1)$	...

For example the circle with center at $(1,2)$ and radius $\sqrt{5}$ can be described by the following parametric equations:

$$x = 1 + \sqrt{5}\cos t,\ y = -2 + \sqrt{5}\sin t,$$

where t is a parameter, $t \in (-\infty; \infty)$. Rewriting the equations in the form: $x - 1 = \sqrt{5}\cos t$, $y + 2 = \sqrt{5}\sin t$, squaring them and summing them we will receive the equation of the circle in xy-coordinates:

$$\begin{cases} x - 1 = \sqrt{5}\cos t, \\ y + 2 = \sqrt{5}\sin t \end{cases} \Leftrightarrow + \begin{cases} (x-1)^2 = 5\cos^2 t, \\ (y+2)^2 = 5\sin^2 t \end{cases}$$

$$(x-1)^2 + (y+2)^2 = 5(\cos^2 t + \sin^2 t) \text{ or}$$

$$(x-1)^2 + (y+2)^2 = 5.$$

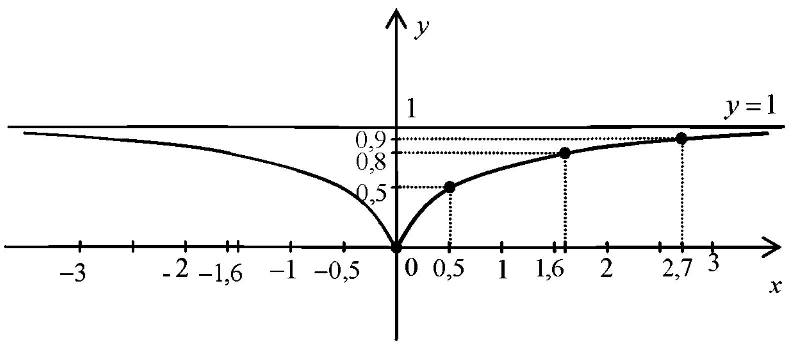
Example 1.18. Sketch the graph of a function $x=\dfrac{t^3}{1+t^2}$, $y=\dfrac{t^2}{1+t^2}$

□ Obviously $D(x)=\mathbb{R}$, $D(y)=\mathbb{R}$. Thus $t\in(-\infty;+\infty)$. The range of $x(t)$ that can be denoted by $E(x)=\mathbb{R}$. Since $y(t)\geq 0\ \forall t\in D(y)$ and $\lim\limits_{t\to\pm\infty} y(t)=\lim\limits_{t\to\pm\infty}\dfrac{t^2}{1+t^2}=1$, $E(y)=[0,1)$. Make a table:

Table 1.5

t	–2	–1	0	1	2	3
x	$-\frac{8}{5}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{8}{5}$	$\frac{27}{10}$
y	$\frac{4}{5}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{9}{10}$

We plot each point $(x(t),y(t))$ and join them. The graph is represented below (pic. 1.15). ■



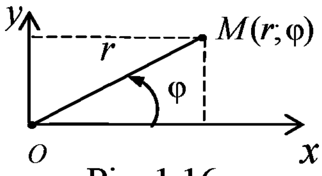
Pic. 1.15

Graphs of functions represented in polar coordinates

Let a point O be the ***pole***, a horizontal half-line Ox – the ***polar axis***.

Then r and φ are ***polar coordinates*** of an arbitrary point M on the plane (pic. 1.16). The ***polar radius*** r is equal to the distance between M and the pole (the length of the segment OM), $r\geq 0$. The ***polar angle*** φ is the angle between segment OM and the polar axis. The angle φ is measured counterclockwise, $\varphi\in\mathbb{R}$.

Let the origin of Cartesian coordinates system coincides with the pole and the positive direction of the x-axis coincides with the polar axis.



Pic. 1.16

Then the relationship between rectangular and polar coordinates of M can be expressed by the following formulas:

$$x = r\cos\varphi,\ \ y = r\sin\varphi;$$

and vice versa:

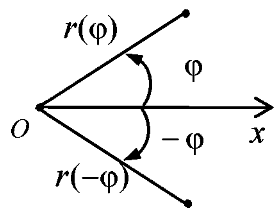
$$r = \sqrt{x^2 + y^2},\quad \varphi = \begin{cases} \arctan\frac{y}{x}, y \ge 0,\ x > 0, \\ \pi + \arctan\frac{y}{x}, y \ge 0,\ x < 0, \\ -\pi + \arctan\frac{y}{x}, y < 0,\ x < 0, \\ \arctan\frac{y}{x}, y < 0,\ x > 0, \\ \frac{\pi}{2}\cdot \mathrm{sign}(y),\ x = 0. \end{cases}$$

The equation $r = r(\varphi)$, $\varphi \in [a;b]$ is called the ***polar equation*** of a curve. The curve exists for all φ: $r(\varphi) \ge 0$.

Example 1.19. Sketch the graph of function $r = a(1+\cos\varphi)$ $a > 0$.

□ Since $-1 \le \cos\varphi \le 1$, $r \ge 0$ for all $\varphi \in \mathbb{R}$. The function $r = a(1+\cos\varphi)$ is a periodic function with the period $T = 2\pi$. Consequently the domain of $r = a(1+\cos\varphi)$ is $D(r) = \{\varphi : -\pi + 2\pi k \le \varphi \le \pi + 2\pi k,\ k \in \mathbb{Z}\}$, the range is $E(r) = [0;\ 2a]$. It can be easy to verify that $r(-\varphi) = r(\varphi)$. Indeed, $r(-\varphi) = a(1+\cos(-\varphi)) = a(1+\cos\varphi) = r(\varphi)$.

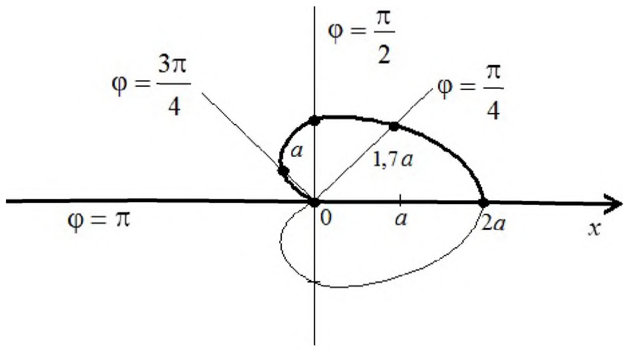
It means that the curve is symmetric about the polar axis (pic. 1.17). Taking into account the fact of symmetry of the graph and periodicity we make the table for $\varphi \in [0;\pi]$.



Pic. 1.17.

Table 1.6

φ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	$2a$	$a\left(1+\frac{\sqrt{2}}{2}\right)\approx 1,$	a	$a\left(1-\frac{\sqrt{2}}{2}\right)\approx 0,3a$	0



Pic.1.18

This curve is called the ***cardioid.*** ■

Example 1.20. Sketch the graph of the function $r=\sin 2\varphi$.

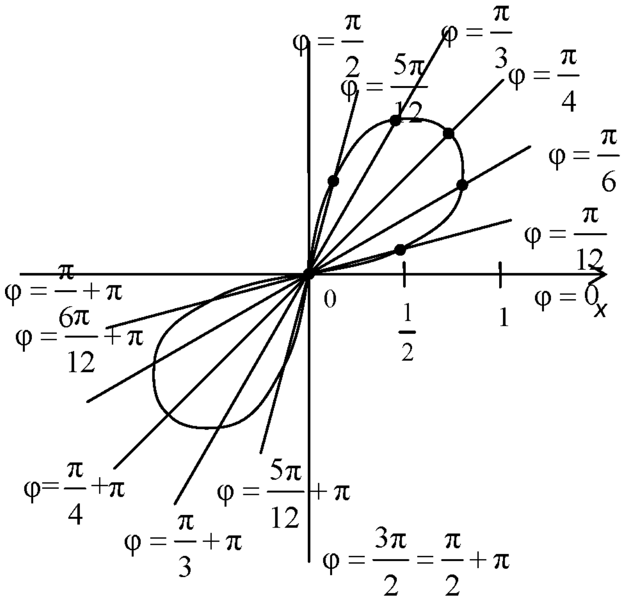
□ The function $\sin 2\varphi$ is a periodic function with the period $T=\frac{2\pi}{2}=\pi$. The given function is defined for $\varphi\in\mathbb{R}$: $\sin 2\varphi\geq 0$. So the domain of $\sin 2\varphi$ is $D(r)=\{\varphi: 2\pi k\leq 2\varphi\leq \pi+2\pi k,\ k\in\mathbb{Z}\}$ or $D(r)=\left\{\varphi: \pi k\leq\varphi\leq\frac{\pi}{2}+\pi k,\ k\in\mathbb{Z}\right\}$.

That means the curve is located on the first and the third coordinate quarters and has two identical parts for $\varphi\in\left[0;\frac{\pi}{2}\right]$ and $\varphi\in\left[\pi;\frac{3\pi}{2}\right]$ because $\sin 2(\varphi+\pi)=\sin 2\varphi$.

Thus, to plot the graph it's enough to consider the interval $\left[0;\frac{\pi}{2}\right]$. ■

Table 1.7.

φ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0



Pic. 1.19

Exercises

1. Find the domain of the functions:

   a) $y = \sqrt{x+5} - \sqrt{-8-x}$;

   b) $y = \sqrt{9-x^2}\arctan\dfrac{1}{x}$;

   c) $y = \arccos\dfrac{2x}{1+x^2}$.

2. Find the range of the functions:

   a) $y = 2^{1/x}$;

   b) $y = \sin x + \cos x$;

   c) $y = \lg(1 - 2\cos x)$.

3. Using graph transformations sketch graphs of functions given below:

   a) $y = |x-3|$;

   b) $y = 2^{1-x}$;

   c) $y = \log_2(x-1)$

   d) $y = \dfrac{2-x}{3-x}$.

CHAPTER 2. LIMITS AND CONTINUITY

2.1. NUMBER SEQUENCES. LIMITS OF NUMBER SEQUENCES

Def.: A function f that maps the set of natural numbers $\mathbb{N}$ into a set X ($f:\mathbb{N}\to X$) is called a ***number sequence***.

As a result we will come to the notation:

$$x_n = f(n),\, n\in\mathbb{N} \quad \text{or} \quad \{x_n\}_{n=1}^{\infty}.$$

Under this concept x_i is the ith term (element) of a sequence, x_n is ***the nth term or the general term*** of a sequence.

Example 2.1: Write out the first four terms of the following sequences:

a) $x_n = \dfrac{(-1)^n}{n},\, n\in\mathbb{N}$;

b) $x_n = \dfrac{2n+1}{n^2},\ n\in\mathbb{N}$;

c) $x_n = \sin\dfrac{\pi n}{2},\, n\in\mathbb{N}$.

□ If $n=1$ $x_1 = \dfrac{(-1)^1}{1} = 1$. In similar fashion we have $x_2 = \dfrac{1}{2}, x_3 = -\dfrac{1}{3},\ x_4 = \dfrac{1}{4}$.

a) $x_1 = \dfrac{2\cdot 1+1}{1^2} = 3,\ x_2 = \dfrac{5}{4}, x_3 = \dfrac{7}{9},\ x_4 = \dfrac{9}{16}$.

b) $x_1 = \sin\dfrac{\pi\cdot 1}{2} = 1,\ x_2 = \sin\pi = 0,\ x_3 = \sin\dfrac{3\pi}{2} = -1,\ x_4 = \sin 2\pi = 0$. ■

Def.: $\{x_n\}_{n=1}^{\infty}$ is ***bounded from above*** if there exists a number A such that $x_n \le A$ for all $n\in\mathbb{N}$.

Def.: $\{x_n\}_{n=1}^{\infty}$ is ***bounded from below*** if there exists a number a such that $x_n \ge a$ for all $n\in\mathbb{N}$.

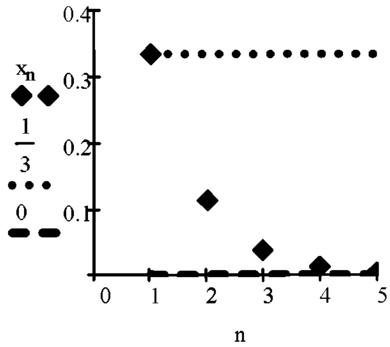
Def.: $\{x_n\}_{n=1}^{\infty}$ is bounded if $\{x_n\}_{n=1}^{\infty}$ is ***bounded*** both from above and from below.

Example 2.2. Determine if $\{x_n\}_{n=1}^{\infty}$ is bounded from below or from above or it is unbounded.

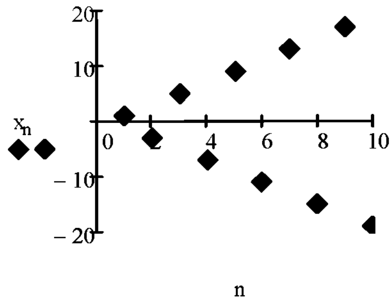
a) $X = \left\{\dfrac{1}{3}, \dfrac{1}{3^2}, \dots, \dfrac{1}{3^n}, \dots\right\}, n\in\mathbb{N}$;

b) $X = \{1, -3, 5, -7, \dots, (-1)^{n+1}(2n-1), \dots\}, n\in\mathbb{N}$.

□ All of the given sequences is determined by listening vales of their elements. As we know a sequence is a particular case of a function. Consequently it can be depicted on the xy coordinate plane (pic. 2.1 *a*–*b*).



a



b

Pic. 2.1

a) As pic. 2.1, a shows all points interpreted as terms of the sequence are inside a strip region bounded by two lines with y-intercepts 0 and $\frac{1}{3}$. It means it can be found two numbers a and A (in the considered case we may put $A=\frac{1}{3}$ and $a=0$) such that $x_n \le A$ and $x_n \ge a$ for all $n \in \mathbb{N}$. Thus in case a is bounded both from above and from below or just bounded.

b) Apparently there are no such a and A that $x_n \le A$ and $x_n \ge a$ hold for all $n \in \mathbb{N}$. Even more $|x_n| > b$ is met for any real $b > 0$. Hence the sequence is unbounded. ■

Note, the numbers a and A are not unique. For example, the sequence $\{x_n\}_{n=1}^{\infty}$ with the range $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}, n \in \mathbb{N}$ is bounded from above. So any number greater than or equal to 1 may be taken for the number A.

Def.: The least number $\tilde{A}$: $x_n \le \tilde{A},\ n \in \mathbb{N}$ is called ***the supremum*** of $\{x_n\}_{n=1}^{\infty}$ or $\tilde{A} = \sup\{x_n\}$.

Def.: The greatest number $\tilde{a}$: $x_n \ge \tilde{a},\ n \in \mathbb{N}$ is called ***the infimum*** of $\{x_n\}_{n=1}^{\infty}$ or $\tilde{a} = \inf\{x_n\}$.

Example 2.3. Find $\sup\{x_n\}$ and $\inf\{x_n\}$, if

a) $x_n = \frac{1}{n},\ n \in \mathbb{N}$.

b) $x_n = n^2 + 1,\ n \in \mathbb{N}$

□ a) $\sup\{x_n\} = 1,\ \inf\{x_n\} = 0$; b) $\sup\{x_n\} = +\infty,\ \inf\{x_n\} = 2$. ■

Note, the supremum and the infimum of a sequence always exist.

Def.: A sequence $\{x_n\}_{n=1}^{\infty}$ is ***increasing*** (***decreasing***) if $x_n \le x_{n+1}$ ($x_n \ge x_{n+1}$) for all $n \in \mathbb{N}$.

Def.: A sequence $\{x_n\}_{n=1}^{\infty}$ is ***strictly increasing*** (***strictly decreasing***) if $x_n < x_{n+1}$ ($x_n > x_{n+1}$) for all $n \in \mathbb{N}$.

Def.: Increasing or decreasing sequences are called ***monotonic*** sequences.

Example 2.4. Determine if $\{x_n\}_{n=1}^{\infty}$ is increasing or decreasing or it is not monotonic.

a) $x_n = 2n+1, n \in \mathbb{N}$.

b) $x_n = \dfrac{(-1)^n}{n}, n \in \mathbb{N}$.

□

a) Since $x_{n+1} = 2(n+1)+1 = 2n+3 > x_n = 2n+1$ $\{x_n\}_{n=1}^{\infty}$ is strictly increasing.

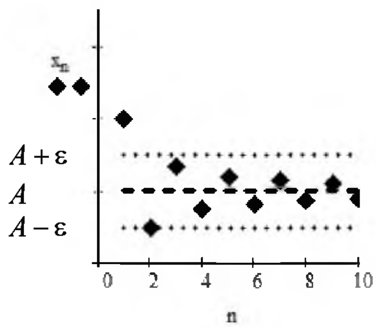
b) Calculate the first three terms of $\{x_n\}_{n=1}^{\infty}$: $x_1 = -1, x_2 = \frac{1}{2}, x_3 = -\frac{1}{3}$. It can be easily shown that there is no any tendency for $\{x_n\}_{n=1}^{\infty}$. Indeed, $x_1 < x_2, x_2 > x_3$. Thus $\{x_n\}_{n=1}^{\infty}$ is neither increasing nor decreasing. ■

Def.: The statement

$$\lim_{n\to\infty} x_n = A$$

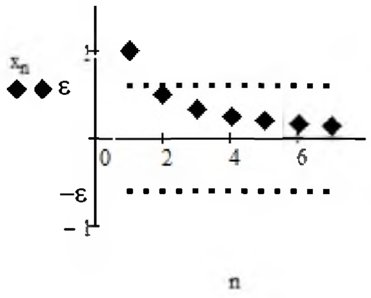
means that for any given $\varepsilon > 0$ there exists a number $\tilde{N} = \tilde{N}(\varepsilon)$ such that $|x_n - A| < \varepsilon$ for all $n \in \mathbb{N}$.

The inequality $|x_n - A| < \varepsilon$ is equivalent to the fact that terms x_n belong to ε-neighborhood of A. Graphically it means that points interpreted as terms x_n are situated inside the strip region shown below (pic. 2.2).



Pic. 2.2

Consider the sequence $x_n = \frac{1}{n}, n \in \mathbb{N}$. The graph of $\{x_n\}_{n=1}^{\infty}$ is depicted below (pic. 2.3). Fix some small $\varepsilon > 0$ and draw a straight line with *y*-intercept ε. For example, let $\varepsilon = 0.6$.



Pic. 2.3

Since all terms of the given sequence are positive it's enough to focus on the part of the region located above the *x*-axis. Terms with subscripts greater than 2 are within the considered region.

And a finite number that is only one element for the chosen ε is outside the region. Similar behavior keeps without changing whatever ε we choose. This demonstrates the fact that limit exists. Moreover the greater n is the closer x_n is to zero. And it's intuitively understood that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Proposition 2.1. *If a sequence has a limit it is unique.*

Theorem 2.1 (The Convergence theorem)

a) *Every increasing sequence* $\{x_n\}_{n=1}^{\infty}$ *that is bounded from above is convergent (there is a finite limit of* $\{x_n\}_{n=1}^{\infty}$*) and*

$$\lim_{n\to\infty} x_n = \sup\{x_n\}.$$

b) *Every decreasing sequence* $\{x_n\}_{n=1}^{\infty}$ *that is bounded from below is convergent (there is a finite limit of* $\{x_n\}_{n=1}^{\infty}$*) and*

$$\lim_{n\to\infty} x_n = \inf\{x_n\}.$$

Using the Convergence theorem it can be proved that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \left[1^{\infty}\right] = e,$$

where e is called ***the Neper's number***, $e = 2{,}18281...$

2.2. LIMITS OF A FUNCTION

It is often necessary to study behavior of a function in a neighborhood of some point. Consider a function $y = f(x)$ which is defined in some neighborhood of a point x_0. Analyzing behavior of f means that we equate the argument x to values which approach x_0 as close as we wish: $x = x_0 \pm 0{,}1$; $x = x_0 \pm 0{,}01$; $x = x_0 \pm 0{,}01$ and so on (this procedure is denoted by the symbols: $x \to x_0$) and then we calculate the corresponding values of the function: $f(x_0 \pm 0{,}1)$; $f(x_0 \pm 0{,}01)$; $f(x_0 \pm 0{,}001)$,.... that may approach the definite number A or not.

Let's consider $f(x) = x^2$, $x_0 = 2$. Then at $x = 2 + 0{,}1$ $f(x) = 2{,}1^2 = 4{,}41$; at $x = 2 + 0{,}01$ $f(x) = 2{,}01^2 = 4{,}0401$; at $x = 2 + 0{,}001$ $f(x) = 2{,}001^2 = 4{,}004001$. In this case the values of the function obviously approach 4. It is said that the function $f(x) = x^2$ has the finite limit 4 at 2. The common notation: $\lim\limits_{x\to 2} x^2 = 4$ or $x^2 \to 4$ if $x \to 2$. This statement will be proved below.

Def.: The statement

$$\lim_{x \to x_0} f(x) = A$$

means that for any given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that $|f(x) - A| < \varepsilon$ whenever $|x - x_0| < \delta$.

Note, that $f(x)$ isn't needed to be equal to A at x_0; in fact, it can be even undefined at x_0. The given above definition can be referred to as the (ε, δ) - definition of a limit or ***Cauchy's definition of a limit***.

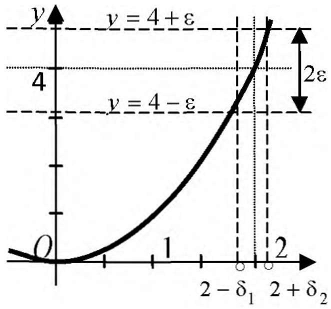
Example 2.5. Prove, that $\lim\limits_{x\to 2} x^2 = 4$.

□ Let's take $\varepsilon > 0$ what we wish. Now, we want $f(x) = x^2$ to differ from 4 by less than ε. In other words, we want $|f(x) - 4| = |x^2 - 4| < \varepsilon$. Solving this inequality, we have

$$|x^2 - 4| < \varepsilon \Leftrightarrow \begin{cases} x^2 - 4 < \varepsilon \\ x^2 - 4 > -\varepsilon \end{cases} \Leftrightarrow \begin{cases} x^2 < 4 + \varepsilon \\ x^2 > 4 - \varepsilon \end{cases} \Leftrightarrow \begin{cases} x < \sqrt{4+\varepsilon} \\ x > \sqrt{4-\varepsilon} \end{cases} \Leftrightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}\ .$$

Since $x \in U(2)$, only positive solutions of the inequality should be considered. We may represent numbers in the left and the right sides of the last inequality in the following form: $\sqrt{4-\varepsilon} = 2 - \delta_1\ (\delta_1 > 0)$, $\sqrt{4+\varepsilon} = 2 + \delta_2\ (\delta_2 > 0)$ (pic. 2.4).

Let δ be $\min\{\delta_1, \delta_2\}$. It is obvious that $\delta = \delta_2 = \sqrt{4+\varepsilon} - 2$ (see pic. 2.4). This is guaranteed if $|x - 2| < \delta$, thus for the considered $\varepsilon > 0$ choosing x within a symmetric neighborhood of 2 with radius δ guarantees that $f(x)$ is within a symmetric neighborhood of 4 with radius ε or $\lim_{x\to 2} x^2 = 4$. ■



Pic. 2.4

Theorem 2.2 (Uniqueness of a limit)

A function $y = f(x)$ has at most one limit.

Theorem 2.3

If a function $y = f(x)$ has a finite limit as $x \to x_0$ then $f(x)$ is locally bounded near x_0.

Remark 2.1

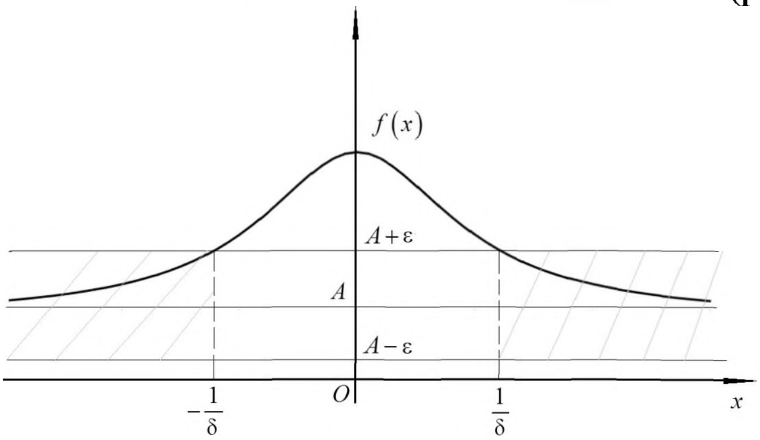
The definition given above is valid only for the case when x_0 and A are finite. If $x_0 = \infty$ the definition can be modified as follows:

$$\lim_{x\to\infty} f(x) = A$$

if for any given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that

$$|f(x) - A| < \varepsilon \text{ whenever } |x| > \frac{1}{\delta}.$$

Graphical interpretation of the considered case is illustrated below (pic. 2.5)

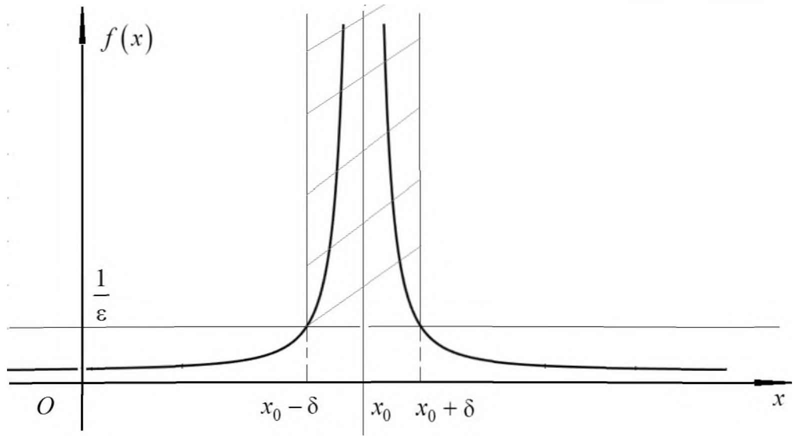


Pic. 2.5

If x_0 is finite and $A=\infty$ the definition can be written in the form:

$$\lim_{x\to x_0} f(x)=\infty$$

if for any given $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)$ such that $|f(x)|>\frac{1}{\varepsilon}$ whenever $|x-x_0|<\delta$ (pic. 2.6).



Pic. 2.6

The negation of existence of a limit can be formulated as follows: there ***exists*** $\varepsilon>0$, such that for ***all*** $\delta>0$, there ***exists*** x, which satisfies $0<|x-x_0|<\delta$, but $|f(x)-A|>\varepsilon$.

Remark 2.2

1. None of the trigonometric functions ($\sin x$, $\cos x$, $\tan x$, $\cot x$) has a limit as $x\to\infty$.

2. The functions $\arcsin x$ and $\arccos x$ don't have limits as $x\to\infty$ because their domains are bounded sets. At the same time

$$\lim_{x\to+\infty}\arctan x=\frac{\pi}{2},\ \lim_{x\to-\infty}\arctan x=-\frac{\pi}{2}.$$

Def.: A function $f(x)$ is called an ***infinitely large*** function, if $\lim_{x\to x_0} f(x)=\infty$

Def.: A function $f(x)$ is called an ***infinitesimal*** function, if $\lim_{x\to x_0} f(x)=0$.

General theorems about limits

Theorem 2.4

$f(x)$ has a finite limit A as x approaches x_0 if and only if there exists the representation $f(x)=A+\alpha(x)$ where $\alpha(x)$ is an infinitesimal function as $x\to x_0$ or $f(x)=A+\alpha(x),\alpha(x)\underset{x\to x_0}{\to}0\Leftrightarrow\lim_{x\to x_0} f(x)=A$.

Theorem 2.5 (connection between an infinitesimal function and an infinitely large function)

a) *if* $\alpha(x)$ *is an infinitesimal function as* $x \to x_0$ *and* $\alpha(x) \neq 0$ *then* $A(x) = \dfrac{1}{\alpha(x)}$ *is an infinitely large function as* $x \to x_0$.

b) *if* $A(x)$ *is an infinitely large function as* $x \to x_0$ *and* $A(x) \neq 0$ *then* $\alpha(x) = \dfrac{1}{A(x)}$ *is an infinitesimal function as* $x \to x_0$.

Theorem 2.6 (limits arithmetic)

Let $f(x)$ *and* $g(x)$ *be defined in a punctured neighborhood of* x_0 ($\overset{o}{U}(x_0) = U(x_0) \setminus \{x_0\}$) *and* $\lim\limits_{x \to x_0} f(x) = A$, $\lim\limits_{x \to x_0} g(x) = B$, *where* A, B *are constant. Then*

a) $\lim\limits_{x \to x_0} \{f(x) \pm g(x)\} = A \pm B$;

b) $\lim\limits_{x \to x_0} \{f(x) \cdot g(x)\} = A \cdot B$;

c) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{A}{B}$, $g(x) \neq 0$.

Corollary

$\lim\limits_{x \to x_0} \{C \cdot f(x)\} = C \cdot \lim\limits_{x \to x_0} f(x)$, $C = \text{const}$.

Theorem 2.7 (limits of a composite function)

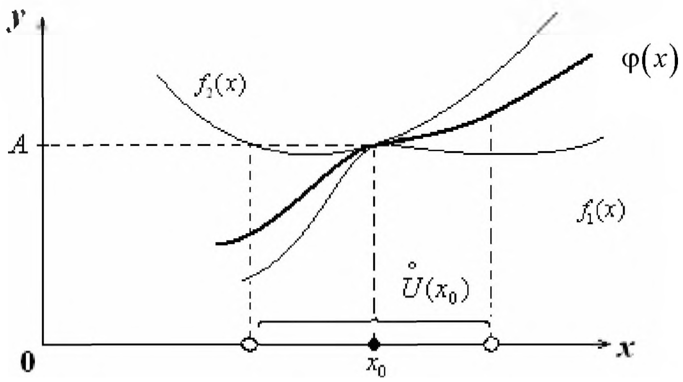
Let y *be a function of* u ($y = f(u)$) *and* u *be a function of* x ($u = g(x)$). $f(u)$ *and* $g(x)$ *are defined in a neighborhood of* u_0 ($U(u_0)$) *and a neighborhood of* x_0 ($U(x_0)$) *respectively. If* $\lim\limits_{x \to x_0} g(x) = u_0$ *and* $\lim\limits_{u \to u_0} f(u) = A$ *then* $f(g(x))$ *has a limit at* x_0 *and* $\lim\limits_{x \to x_0} f(g(x)) = A$.

Theorem 2.8

Let $f_1(x)$ *and* $f_2(x)$ *be defined in* $U(x_0)$ *and* $\lim\limits_{x \to x_0} f_1(x) = A$, $\lim\limits_{x \to x_0} f_2(x) = B$. *If* $f_1(x) < f_2(x)$ *or* $f_1(x) \leq f_2(x)$ *hold for* $\forall x \in U(x_0)$ *then* $A \leq B$.

Theorem 2.9 (Sandwich theorem)

Let $f_1(x)$, $f_2(x)$ *and* $\varphi(x)$ *be defined in* $\overset{o}{U}(x_0)$ *and* $\lim\limits_{x \to x_0} f_1(x) = A$, $\lim\limits_{x \to x_0} f_2(x) = A$. *Suppose,* $f_1(x) \leq \varphi(x) \leq f_2(x)$ *holds for* $\forall x \in \overset{o}{U}(x_0)$. *Then* $\lim\limits_{x \to x_0} \varphi(x) = A$ (pic. 2.7).



Pic. 2.7

Theorem 2.10

If $f_1(x)$ is an elementary function, defined in $U(x_0)$, where $x_0 \in D_f$, then $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

Thus, for example, $\lim\limits_{x\to x_0} e^x = e^{x_0}$ or $\lim\limits_{x\to x_0} \sin x = \sin x_0$ are valid for any real x_0.

Properties of infinitesimal functions

Let $\alpha(x)$ and $\beta(x)$ be infinitesimal functions as $x \to x_0$: $\lim\limits_{x\to x_0} \alpha(x) = 0,\ \lim\limits_{x\to x_0} \beta(x) = 0$.

1. $\lim\limits_{x\to x_0} \{\alpha(x) \pm \beta(x)\} = 0$;
2. $\lim\limits_{x\to x_0} \alpha(x) \cdot f(x) = 0$, where $\lim\limits_{x\to x_0} f(x) = A$;
3. $\lim\limits_{x\to x_0} \dfrac{\alpha(x)}{f(x)} = 0$, where $\lim\limits_{x\to x_0} f(x) = A,\ A \neq 0$.

We can visualize the given above properties as the scheme:

$$0 \pm 0 = 0,$$
$$0 \cdot A = 0, A = \text{const},$$
$$\frac{0}{A} = 0,\ A = \text{const},\ A \neq 0.$$

Properties of infinitely large functions

Let $F(x)$ and $G(x)$ be infinitesimal functions as $x \to x_0$: $\lim\limits_{x\to x_0} F(x) = \infty,\ \lim\limits_{x\to x_0} G(x) = \infty$.

1. $\lim\limits_{x\to x_0}\{F(x)+G(x)\}=\infty$;
2. $\lim\limits_{x\to x_0}\{F(x)+f(x)\}=\infty$, where $\lim\limits_{x\to x_0} f(x)=A,\ A=\text{const}$;
3. $\lim\limits_{x\to x_0} F(x)\cdot f(x)=\infty$, where $\lim\limits_{x\to x_0} f(x)=A,\ A\neq 0$.
4. $\lim\limits_{x\to x_0} F(x)\cdot G(x)=\infty$.

Visualization of the properties are:

$$\infty+\infty=\infty,$$
$$\infty+A=\infty, A=\text{const},$$
$$\infty\cdot A=\infty, A=\text{const},$$
$$\infty\cdot\infty=\infty.$$

Example 2.6. Find $\lim\limits_{x\to 7}\frac{3x+5}{x-5}$.

□ The rational fraction $\frac{3x+5}{x-5}$ is an elementary function that is defined at the limit point 7. So $\lim\limits_{x\to 7}\frac{3x+5}{x-5}=\left(\frac{3x+5}{x-5}\right)\Big|_7=\frac{3\cdot 7+5}{7-5}=\frac{26}{2}=13$. ■

Example 2.7. Find $\lim\limits_{x\to 0}2^x\left(x^2-4\right)$.

□ The elementary function $2^x\left(x^2-4\right)$ is defined at 0. Consequently, $\lim\limits_{x\to 0}2^x\left(x^2-4\right)=2^0\left(0^2-4\right)=1\cdot(-4)=-4$. ■

Example 2.8. Find $\lim\limits_{x\to 5}\frac{3x+5}{x-5}$.

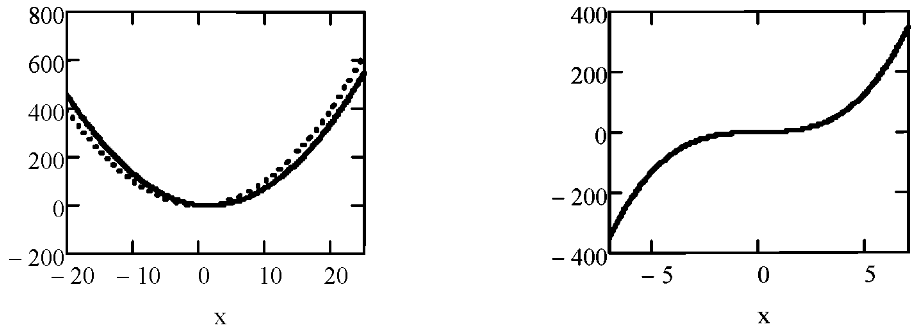
□ Inspite of example 2 the fraction $\frac{3x+5}{x-5}$ isn't defined at 5. To give the answer we should represent $\frac{3x+5}{x-5}$ as a product: $\frac{3x+5}{x-5}=(3x+5)\cdot\frac{1}{x-5}$ and find limits of each factor. Then, $\lim\limits_{x\to 5}(3x+5)=3\cdot 5+5=20$.

The function $x-5$ is an infinitesimal one as $x\to 5$, so according to theorem 2.5 $\frac{1}{x-5}$ is an infinitely large function: $\lim\limits_{x\to 5}\frac{1}{x-5}=\infty$. Then the third property of infinitely large functions can be applied: $\lim\limits_{x\to 5}\frac{3x+5}{x-5}=\lim\limits_{x\to 5}(3x+5)\cdot\frac{1}{x-5}=\infty$. ■

Example 2.9. Find $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}$.

□ Due to non-existence of a limit of $\sin x$ as $x\to\infty$ we can't calculate the limit as a quotient of two functions: $\sin x$ and x. To get the answer we should rewrite the function $\dfrac{\sin x}{x}$ as a product: $\dfrac{\sin x}{x}=\sin x\cdot\dfrac{1}{x}$. Since $|\sin x|\le 1$ for any $x\in\mathbb{R}$, the first factor $\sin x$ is a bounded function. According to theorem 2.5 the second factor $\dfrac{1}{x}$ is an infinitesimal function as $x\to\infty$. So we can apply the third property of infinitesimal functions and receive: $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\lim\limits_{x\to\infty}\sin x\cdot\dfrac{1}{x}=0$. ■

Let's consider the problem of calculating a limit of $\dfrac{x^2-3x+1}{x^3+x}$ as $x\to\infty$: $\lim\limits_{x\to\infty}\dfrac{x^2-3x+1}{x^3+x}$. The function $\dfrac{x^2-3x+1}{x^3+x}$ is a rational fraction, whose numerator and denominator are polynomials. Graphs of the polynomials are given below (pic. 2.8). The solid and dash lines are used for the graphs of x^2-3x+1 and x^2 respectively (pic. 2.8, *a*). The graph of x^3+x is depicted in pic. 2.8, *b*.



a *b*

Pic. 2.8

As shown in pic. 2.8 the numerator approach $+\infty$ as $x\to+\infty$ or $x\to-\infty$ and the denominator approach $-\infty$ as $x\to-\infty$ and $+\infty$ as $x\to+\infty$. If we omit sign of infinity we will get the expression $\left[\dfrac{\infty}{\infty}\right]$. This expression is called the ***indeterminate form***. The word indeterminate is used because a further analyses is necessary to conclude whether a limit exists or not.

To calculate the limit we should carry out some algebraic transformations of the given function. In this case, we say, that we investigate the indeterminate form. Other examples of indeterminate forms are: $\left[\dfrac{0}{0}\right]$, $[0\cdot\infty]$, $[\infty-\infty]$, $[1^\infty]$, $[0^0]$, $[\infty^0]$.

Firstly we should identify the highest exponent of x in the denominator and divide both the numerator and the denominator by it. Then the limit of all remaining terms should be taken.

$$\lim_{x\to\infty}\frac{x^2-3x+1}{x^3+x}=\lim_{x\to\infty}\frac{x^3\left(\frac{1}{x}-\frac{3}{x^2}+\frac{1}{x^3}\right)}{x^3\left(1+\frac{1}{x^2}\right)}=\left|\begin{array}{l}\frac{1}{x}\underset{x\to\infty}{\to}0,\\ \frac{1}{x^2}\underset{x\to\infty}{\to}0,\\ \frac{1}{x^3}\underset{x\to\infty}{\to}0\end{array}\right|=\left[\frac{0-0+0}{1+0}\right]=0.$$

In a similar fashion we can show

$$\lim_{x\to\infty}\frac{7x^3+8x^2+3}{5x^3+2x+1}=\frac{7}{5};\quad \lim_{x\to\infty}\frac{10-x-2x^2+3x^6}{4x^3-5x-6}=\infty.$$

According to these results it leads us to the rule:

$$\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)}=\lim_{x\to\infty}\frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n}{b_0x^m+b_1x^{m-1}+b_2x^{m-2}+\ldots+b_{m-1}x+b_m}=\begin{cases}0, & \text{if } n<m\\ \frac{a_0}{b_0}, & \text{if } n=m,\\ \infty, & \text{if } n>m.\end{cases}$$

The limit $\lim_{x\to\infty}\frac{P_n(x)}{Q_m(x)}$ is referred to as ***the third remarkable limit***.

Example 2.10. Find $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$.

□ We are dealing with one more indeterminate form $[\infty-\infty]$. In the considered case the given function involves radicals, so to find the limit we should multiply the function $\sqrt{x^2+1}-\sqrt{x^2-1}$ by it's conjugate, that is $\sqrt{x^2+1}+\sqrt{x^2-1}$, and then divide it by the same expression $\sqrt{x^2+1}+\sqrt{x^2-1}$. It should be noted that multiplying by the conjugate is equivalent to applying the "difference of squares" formula $(a-b)(a+b)=a^2-b^2$. That implies eliminating radicals. Moreover, the conjugate $\sqrt{x^2+1}+\sqrt{x^2-1}\to\infty$ as $x\to\infty$.

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)}=$$

$$= \lim_{x \to \infty} \frac{2}{\left(\sqrt{x^2+1}+\sqrt{x^2-1}\right)} = \left[\frac{2}{\infty}\right] = 0 . ■$$

Example 2.11. Find $\lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1}$.

□ Let's evaluate first the numerator $x^2 - 4x + 3$ and the denominator $x - 1$ at $x = 1$: $x^2 - 4x + 3\big|_{x=1} = 1 - 4 + 3 = 0$ and $x - 1\big|_{x-1} = 1 - 1 = 0$. It leads to the indeterminate form $\left[\frac{0}{0}\right]$. Nevertheless the given function is a rational function our strategy differs from the previous one. We will simplify the fraction by factorizing both the numerator and the denominator. Since $x^2 - 4x + 3 = (x-1)(x-3)$,

$$\frac{x^2 - 4x + 3}{x - 1} = \frac{(x-1)(x-3)}{(x-1)} = x - 3 .$$

This transformation holds for all values of x: $x \neq 1$. Thus, we get

$$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x-3)}{(x-1)} = 1 - 3 = -2 . ■$$

Example 2.12. Find $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 6x + 9}$.

□ Following the strategy given above, substituting 3 for x in the fraction $\frac{x^2 - 4x + 3}{x^2 - 6x + 9}$ leads to the indeterminate form $\left[\frac{0}{0}\right]$. So factorizing gives us:

$$\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 6x + 9} = \lim_{x \to 3} \frac{(x-1)(x-3)}{(x-3)^2} .$$

The obtained fraction can be simplified by cancelling the common factor $(x-3)$: $\lim_{x \to 3} \frac{(x-1)(x-3)}{(x-3)^2} = \lim_{x \to 3} \frac{(x-1)}{(x-3)} = \left[\frac{2}{0}\right] = \infty$. ■

Example 2.13. Find $\lim_{x \to 2} \frac{\sqrt{x+2} - \sqrt{6-x}}{x^2 - 4}$.

□ Substituting 2 for x in the given function $\frac{\sqrt{x+2} - \sqrt{6-x}}{x^2 - 4}$ leads to the indeterminate form $\left[\frac{0}{0}\right]$. Moreover the numerator $\sqrt{x+2} - \sqrt{6-x}$ is an irrational function involving square roots. So firstly we should multiply the function by the conjugate of $\sqrt{x+2}-\sqrt{6-x}$: $\sqrt{x+2}+\sqrt{6-x}$ and then divide it by the same expression. Further we should factorize the denominator $x^2-4=(x-2)(x+2)$. Thus,

$$\lim_{x\to 2}\frac{\sqrt{x+2}-\sqrt{6-x}}{x^2-4}=\lim_{x\to 2}\frac{\left(\sqrt{x+2}-\sqrt{6-x}\right)\left(\sqrt{x+2}+\sqrt{6-x}\right)}{(x-2)(x+2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}=$$

$$=\lim_{x\to 2}\frac{x+2-(6-x)}{(x-2)(x+2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}=\lim_{x\to 2}\frac{2(x-2)}{(x-2)(x+2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}=$$

$$=\lim_{x\to 2}\frac{2}{(x+2)\left(\sqrt{x+2}+\sqrt{6-x}\right)}=\frac{2}{4\cdot 4}=\frac{1}{8}.\blacksquare$$

Example 2.14. Find $\lim\limits_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$.

□ Inspite of the previous cases when given functions were rational fractions, whose numerator and denominator were polynomials, the considered fraction involves radicals. Moreover, after substituting 1 for x in $\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ we will have the indeterminate form $\left[\frac{0}{0}\right]$. We can convert $\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ into a rational fraction by means of substitution. For this reason let's identify the highest exponent of x in the numerator and in the denominator. They are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. Then find the least common multiply of 2 and 3. It's 6. So we should introduce a new variable t such that $x=t^6$. Thus, we have

$$\lim_{x\to 1}\frac{\sqrt{x}-1}{\sqrt[3]{x}-1}=\left|\begin{matrix}x=t^6,\\ x\to 1\Rightarrow t\to 1\end{matrix}\right|=\lim_{x\to 1}\frac{t^3-1}{t^2-1}=\lim_{x\to 1}\frac{(t-1)\left(t^2+t+1\right)}{(t-1)(t+1)}=$$

$$=\lim_{t\to 1}\frac{t^2+t+1}{t+1}=\frac{1^2+1+1}{1+1}=\frac{3}{2}.\blacksquare$$

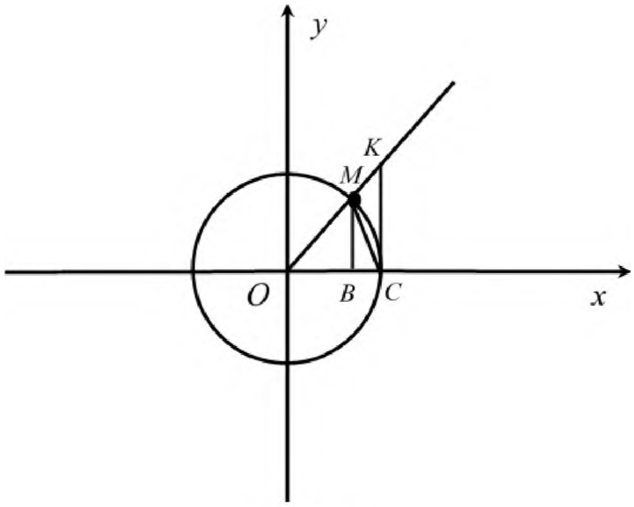
We can single out the special group of limits. These limits are called ***remarkable limits***. One of them so called the third remarkable limit is given above.

Also it can be proved that

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\left[\frac{0}{0}\right]=1.$$

It is called ***the first remarkable limit.***

To prove this fact let's make use of the illustration in pic. 2.9. The proof will be run for two cases: $x \ge 0$ and $x < 0$. Let's start with the case when $x \ge 0$. Plot the unit circle and take a point M on the arc of the circle located in the first coordinate quadrant. Draw two perpendiculars from the points M and C to the x-axis. Then consider ΔKOC, ΔMOC and the sector MOC. Denote the angle $\angle MOC$ as x.



Pic. 2.9

Calculating the area of $\Delta KOC, \Delta MOC$ and the sector MOC, we get $S_{\Delta KOC} = \frac{1}{2} KC \cdot OC = \frac{1}{2} \tan x$, $OC = 1$ as a radius of the unit circle, $\tan x = \frac{KC}{OC}$;

$S_{\Delta MOC} = \frac{1}{2} MB \cdot OC = \frac{1}{2} \sin x$, $\sin x = \frac{MB}{OM}$, $OM = 1$ as a radius of the unit circle;

$S_{\sec MOC} = \frac{1}{2} r^2 x = \frac{1}{2} x$, $r = 1$ as a radius of the unit circle.

As it shown in pic. 2.9 $S_{\Delta MOC} \le S_{\sec MOC} \le S_{\Delta KOC} \Rightarrow \frac{1}{2} \sin x \le \frac{1}{2} x \le \frac{1}{2} \tan x$.

Dividing all parts of the inequality by $\sin x$ leads to $1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$. Taking reciprocal of each part and applying the Sandwich theorem we have $\cos x \le \frac{\sin x}{x} \le 1 \Rightarrow \Rightarrow \lim\limits_{x \to 0} \cos x = 1$, $\lim\limits_{x \to 0} 1 = 1 \Rightarrow \lim\limits_{\substack{x \to 0 \\ (x \ge 0)}} \frac{\sin x}{x} = 1$.

Now let $x < 0$. Make the substitution $x = -t$, then $t > 0$. So

$$\lim_{\substack{x \to 0 \\ (x<0)}} \frac{\sin x}{x} = \left| x = -t \Rightarrow t = -x \right| = \lim_{\substack{t \to 0 \\ (t>0)}} \frac{\sin(-t)}{-t} = \lim_{\substack{t \to 0 \\ (t>0)}} \frac{-\sin(t)}{-t} = \lim_{\substack{t \to 0 \\ (t>0)}} \frac{\sin(t)}{t} = 1,$$

that was to be shown.

Example 2.15. Find $\lim\limits_{x \to 0} \frac{\tan(x)}{x}$.

□ Representing $\tan x$ as $\dfrac{\sin x}{\cos x}$, we get

$$\lim_{x\to 0}\frac{\tan x}{x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=1\cdot\frac{1}{1}=1.\blacksquare$$

Example 2.16. Find $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2/2}$.

□ According to the Double angle formula $1-\cos x=2\sin^2\dfrac{x}{2}$. Then

$$\lim_{x\to 0}\frac{1-\cos x}{x^2/2}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2/2}=\lim_{x\to 0}\frac{2\sin\frac{x}{2}}{x}\cdot\frac{\sin\frac{x}{2}}{x/2}=\lim_{x\to 0}\frac{2\sin\frac{x}{2}}{2\cdot x/2}\cdot\frac{\sin\frac{x}{2}}{x/2}=$$

$$=\frac{2}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{x/2}\cdot\frac{\sin\frac{x}{2}}{x/2}=1\cdot 1=1.\blacksquare$$

Also it can be proved in a similar way that $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$; $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$; $\lim\limits_{x\to 0}\dfrac{(1+x)^\alpha-1}{\alpha x}=1$; $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x}-1}{\frac{x}{n}}=1$; $\lim\limits_{x\to 0}\dfrac{\arcsin x}{x}=1$; $\lim\limits_{x\to 0}\dfrac{\arctan x}{x}=1$.

Comparison of infinitesimals. Big O and little o notations

Suppose, $\alpha(x)$ and $\beta(x)$ are infinitesimal functions or infinitesimals as $x\to x_0$, i.e. $\lim\limits_{x\to x_0}\alpha(x)=0$ and $\lim\limits_{x\to x_0}\beta(x)=0$.

Def.: We say that $\alpha(x)$ is an ***infinitesimal of the same order*** (the same order of smallness) as $\beta(x)$ as $x\to x_0$ if $\lim\limits_{x\to x_0}\dfrac{\alpha(x)}{\beta(x)}=k,\ 0<|k|<\infty$.

It can be written that $\alpha(x)=O(\beta(x)),\ x\to x_0$ and $\beta(x)=O(\alpha(x)),\ x\to x_0$.

Def.: We say that $\alpha(x)$ is an ***infinitesimal of higher order*** than $\beta(x)$ as $x\to x_0$, if $\lim\limits_{x\to x_0}\dfrac{\alpha(x)}{\beta(x)}=0$.

In this case $\alpha(x)=o(\beta(x)),\ x\to x_0$.

Def.: We say that $\alpha(x)$ and $\beta(x)$ are ***equivalent infinitesimals*** as $x\to x_0$, if $\lim\limits_{x\to x_0}\dfrac{\alpha(x)}{\beta(x)}=1$.

Remark 2.3
x_0 can be a constant, $\pm\infty$.

Example 2.17. Let $\alpha(x)=1-x^2$ and $\beta(x)=1-x$. Both functions are infinitesimals as $x\to 1$. Then, $\lim\limits_{x\to 1}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to 1}\frac{1-x^2}{1-x}=2\neq 0$. So $\alpha(x)$ and $\beta(x)$ are infinitesimals of the same order.

Example 2.18. Let $\alpha(x)=x^3$ and $\beta(x)=x^2$. Both functions are infinitesimals as $x\to 0$. Then, $\lim\limits_{x\to 1}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to 1}\frac{x^3}{x^2}=0$. Thus, $\alpha(x)$ is an infinitesimal of higher order than $\beta(x)$.

Example 2.19. Let $\alpha(x)=\sin x$ and $\beta(x)=x$. Both functions are infinitesimals as $x\to 0$. Then, $\lim\limits_{x\to 1}\frac{\alpha(x)}{\beta(x)}=\lim\limits_{x\to 1}\frac{\sin x}{x}=1$. Hence, $\alpha(x)$ and $\beta(x)$ are equivalent infinitesimals.

Summarizing the above, we can make a list of equivalent infinitesimals:

$\sin x\sim x$ as $x\to 0$,
$\tan x\sim x$ as $x\to 0$,
$1-\cos x\sim\frac{x^2}{2}$ as $x\to 0$,
$\arcsin x\sim x$ as $x\to 0$,
$\arctan x\sim x$ as $x\to 0$,
$e^x-1\sim x$ as $x\to 0$,
$\ln(1+x)\sim x$ as $x\to 0$,
$(1+x)^\alpha-1\sim\alpha x$ as $x\to 0$,
$\sqrt[n]{1+x}-1\sim\frac{x}{n}$ as $x\to 0$.

Proposition 2.2
If $f_1(x)\sim\varphi_1(x)$ *and* $f_2(x)\sim\varphi_2(x)$ *as* $x\to x_0$, *then*

$$\lim_{x\to x_0} f_1(x)\varphi_1(x)=\lim_{x\to x_0} f_2(x)\varphi_2(x),\ \lim_{x\to x_0}\frac{f_1(x)}{\varphi_1(x)}=\lim_{x\to x_0}\frac{f_2(x)}{\varphi_2(x)}.$$

Thus, equivalent infinitesimals can be replaced each other.

Example 2.20. Find $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 5x}$.

□ $\lim\limits_{x\to 0}\frac{\sin 3x}{\sin 5x}=\left[\frac{0}{0}\right]=\left|\begin{matrix}\sin 3x\sim 3x, x\to 0\\ \sin 5x\sim 5x,\ x\to 0\end{matrix}\right|=\lim\limits_{x\to 0}\frac{3x}{5x}=\frac{3}{5}$. ■

Example 2.21. Find $\lim\limits_{x\to 0}\frac{\sin x^6}{\sin^5 x}$.

□ $\lim\limits_{x\to 0}\dfrac{\sin x^6}{\sin^5 x}=\left[\dfrac{0}{0}\right]=\left|\begin{matrix}\sin x^6\sim x^6,\ x\to 0\\ \sin^5 x\sim x^5,\ x\to 0\end{matrix}\right|=\lim\limits_{x\to 0}\dfrac{x^6}{x^5}=\lim\limits_{x\to 0}x=0.$ ■

Example 2.22. Find $\lim\limits_{x\to 0}\dfrac{\arctan 5x^2}{1-\cos 3x}$.

□ $\lim\limits_{x\to 0}\dfrac{\arctan 5x^2}{1-\cos 3x}=\left[\dfrac{0}{0}\right]=\left|\begin{matrix}\arctan 5x^2\sim 5x^2,\ x\to 0\\ 1-\cos 3x\sim\dfrac{(3x)^2}{2},x\to 0\end{matrix}\right|=\lim\limits_{x\to 0}\dfrac{5x^2}{\frac{9}{2}x^2}=\dfrac{5\cdot 2}{9}=\dfrac{10}{9}.$ ■

Example 2.23. Find $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

□ $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{\frac{\sin x}{\cos x}-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{\cos x\cdot x^3}=$

$\lim\limits_{x\to 0}\dfrac{1}{\cos x}\cdot\lim\limits_{x\to 0}\dfrac{\sin x(1-\cos x)}{x^3}=\left|\begin{matrix}\sin x\sim x,\ x\to 0\\ 1-\cos x\sim\dfrac{x^2}{2},x\to 0\end{matrix}\right|=1\cdot\lim\limits_{x\to 0}\dfrac{x\cdot\frac{x^2}{2}}{x^3}=\dfrac{1}{2}.$

The limits

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\left[1^\infty\right]=e \quad\text{or}\quad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=\left[1^\infty\right]=e$$

are referred to as the ***second remarkable limit***.

In the previous subsection when we observed number sequences we mentioned that $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e$. Using this fact, let's show that $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$.

The problem can be split into two problems: $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^x=e$ and $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^x=e$.

Let $x\to+\infty$. Then for every positive x we can put that $n\le x<n+1$, where $n=[x]$ is the integer part of x. So $\dfrac{1}{n+1}<\dfrac{1}{x}\le\dfrac{1}{n}\Rightarrow 1+\dfrac{1}{n+1}<1+\dfrac{1}{x}\le 1+\dfrac{1}{n}$.

Thus,

$$\left(1+\frac{1}{n+1}\right)^n<\left(1+\frac{1}{x}\right)^x\le\left(1+\frac{1}{n}\right)^{n+1}.$$

Moreover,

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^n=\frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n+1}\right)}=\frac{e}{1}=e;$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e\cdot 1=e.$$

By the Sandwich theorem $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=e$.

Let $x\to-\infty$. Using the substitution $t=-x$, we get

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x=\left|t=-x\Rightarrow x=-t\right|=\lim_{-t\to-\infty}\left(1-\frac{1}{t}\right)^{-t}=\lim_{t\to+\infty}\left(1+\left(-\frac{1}{t}\right)\right)^{-t}=e.$$

Remark 2.4

We should focus on the fact that the term $\frac{1}{n}$ and the exponent n involved in $\left(1+\frac{1}{n}\right)^n$ as well as $\frac{1}{x}$ and x in $\left(1+\frac{1}{x}\right)^x$ are mutually inverse values, i.e. their product must be equal to 1: for example, $\frac{1}{n}\cdot n=1$.

Example 2.24. Find $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^x$.

□ We deal with the indeterminate form $\left[1^\infty\right]$. So to calculate the limit we should use the second remarkable limit. But first we need to modify the function $\left(1+\frac{3}{x}\right)^x$ making the substitution $\frac{3}{x}=\frac{1}{\alpha}$:

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x=\left[1^\infty\right]=\left|\begin{array}{l}\frac{3}{x}=\frac{1}{\alpha}\\ x=3\alpha\\ x\to\infty\Rightarrow\alpha\to\infty\end{array}\right|=\lim_{\alpha\to\infty}\left(1+\frac{1}{\alpha}\right)^{3\alpha}=\lim_{\alpha\to\infty}\left[\left(1+\frac{1}{\alpha}\right)^\alpha\right]^3=e^3.\ \blacksquare$$

Example 2.25. Find $\lim\limits_{x\to\infty}\left(\frac{3x+1}{3x}\right)^x$.

□ Dividing each term in the numerator by the denominator, we get $\lim\limits_{x\to\infty}\left(\frac{3x+1}{3x}\right)^x=\lim\limits_{x\to\infty}\left(1+\frac{1}{3x}\right)^x$. Further we can follow the strategy given in the previous example.

$$\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^x=\left[1^\infty\right]=\left|\begin{array}{l}\frac{1}{3x}=\frac{1}{\alpha}\\ x=\frac{\alpha}{3}\\ x\to\infty\Rightarrow\alpha\to\infty\end{array}\right|=\lim_{\alpha\to\infty}\left(1+\frac{1}{\alpha}\right)^{\frac{\alpha}{3}}=\lim_{\alpha\to\infty}\left[\left(1+\frac{1}{\alpha}\right)^{\alpha}\right]^{\frac{1}{3}}=e^{\frac{1}{3}}.\blacksquare$$

Example 2.26. Find $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$.

□ To get the answer we should reduce the problem to the second form of the second remarkable limit: $\lim\limits_{\alpha\to 0}(1+\alpha)^{\frac{1}{\alpha}}=e$. For this reason transformation of the exponent $\frac{1}{x}$ into $\frac{1}{\sin x}\cdot\frac{\sin x}{x}$ by multiplying and dividing the exponent by $\sin x$ is offered. Then

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}=\left[1^\infty\right]=\lim_{x\to 0}\left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}}=\lim_{x\to 0}e^{\frac{\sin x}{x}}=e^{\lim\limits_{x\to 0}\frac{\sin x}{x}}=e.$$

Here we applied the theorem about the limit of a composite function and the fact, that the exponential function $e^{(\cdot)}$ is continuous.■

Example 2.27. Find $\lim\limits_{x\to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{1}{\arctan 4x}}$.

□ $$\lim_{x\to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{1}{\arctan 4x}}=\left[1^\infty\right]=\left(\frac{\{1-3x\}+\{3x+2x\}}{1-3x}\right)^{\frac{1}{\arctan 4x}}=\lim_{x\to 0}\left(1+\frac{5x}{1-3x}\right)^{\frac{1}{\arctan 4x}}=$$

$$=\lim_{x\to 0}\left(1+\frac{5x}{1-3x}\right)^{\frac{1-3x}{5x}\cdot\frac{5x}{1-3x}\cdot\frac{1}{\arctan 4x}}==\lim_{x\to 0}\left[\left(1+\frac{5x}{1-3x}\right)^{\frac{1-3x}{5x}}\right]^{\frac{5x}{1-3x}\cdot\frac{1}{\arctan 4x}}=\lim_{x\to 0}e^{\frac{5x}{1-3x}\cdot\frac{1}{4x}}=e^{\frac{5}{4}}.\ \blacksquare$$

Remark 2.5

There is the definition of a limit in terms of sequences (***Heine's definition of a limit***). It states that $f(x)$ has a limit at x_0 or $\lim\limits_{x\to x_0}f(x)=A$ if for every sequence $\{x_n\}_{n=1}^{\infty}$ that approaches x_0, the sequence of the corresponding values $\{f(x_n)\}_{n=1}^{\infty}$ approaches A.

Using this form of the definition it can be enough easy to prove that $\lim\limits_{x\to 0}\sin\frac{1}{x}$ doesn't exist. Consider two sequences: $x_n=\pi n, x_n'=\frac{\pi}{2}+2\pi n, n\in\mathbb{N}$ and find the sequences of the corresponding values: $f(x_n)=\sin\frac{1}{1/\pi n}=\sin\pi n,\ f(x_n')=\sin\left(\frac{\pi}{2}+\pi n\right)$.
It's obvious that $x_n\to 0$, $f(x_n)\to 0$, $n\to\infty$ while $x_n'\to 0$, $f(x_n')\to 1$, $n\to\infty$. By Heine's definition whatever sequence $\{x_n\}_{n=1}^{\infty}$ we take the limit of $\{f(x_n)\}_{n=1}^{\infty}$ must exist and be equal to the same value. In the considered case $\lim\limits_{n\to\infty}f(x_n)\neq\lim\limits_{n\to\infty}f(x_n')$, so the given limit doesn't exist.

Exercises

In the exercises a is a student's number, b is the last numeral in a group number, m is a natural number that can be considered as a parameter:
1. Find $\lim\limits_{n\to\infty}\frac{a^n+3^n}{a^n-3^n}$; 2. Find $\lim\limits_{x\to a}\frac{\cos x-\cos a}{x-a}$; 3. Find $\lim\limits_{x\to 0}\frac{1-\sqrt{\cos x}}{x^3+ax^2}$;

4. Find $\lim\limits_{x\to\infty}\left(\frac{x-a}{x+b}\right)^{x+2}$; 5. Find $\lim\limits_{x\to\infty}\frac{ax^2-b}{bx^2+ax}$; 6. Find $\lim\limits_{x\to 0}\frac{e^{ax}-1}{\sin bx}$;

7. Find $\lim\limits_{x\to 0}\frac{\ln(1+\sin ax)}{\arctan bx}$; 9. Find $\lim\limits_{x\to -1}\frac{mx^3}{x^3-m}$; 10. Find $\lim\limits_{x\to 0}m^{\frac{x}{2}}\cdot\ln(1+x)^m$;

11. Find $\lim\limits_{x\to m}\frac{x^2-m^2}{x^2-(m+1)x+m}$; 12. Find $\lim\limits_{x\to 1}\frac{x^2-m^2}{x^2-(m+1)x+m}$;

13. Find $\lim\limits_{x\to -m}\frac{x^2-m^2}{x^2-(m+1)x+m}$; 14. Find $\lim\limits_{x\to n}\frac{x^3-mx^2}{x^2-mx}$;

15. Find $\lim\limits_{x\to +\infty}(\sqrt{x+2m}-\sqrt{x+m})$; 16. Find $\lim\limits_{x\to m^2}\frac{\sqrt{x}-m}{x-m^2}$;

17. Find $\lim\limits_{x\to n}\frac{x-\sqrt{2x^2-m^2}}{x^2-m^2}$.

2.3. CONTINUITY OF A FUNCTION

In the previous section the concept of a limit of $f(x)$ as $x\to x_0$ was stated under the condition $x\neq x_0$. And what is more, the fact of existing $f(x_0)$ was ignored.

Now let $f(x_0)$ exist and $\lim\limits_{x\to x_0}f(x)=f(x_0)$.

Def.: $f(x)$ is said to be ***continuous*** at x_0 , if

a. $f(x_0)$ exists;

b. $\lim\limits_{x \to x_0} f(x)$ exists;

c. $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Remark 2.6

According to the definition given above and the fact that $\lim\limits_{x \to x_0} x = x_0$ we can put

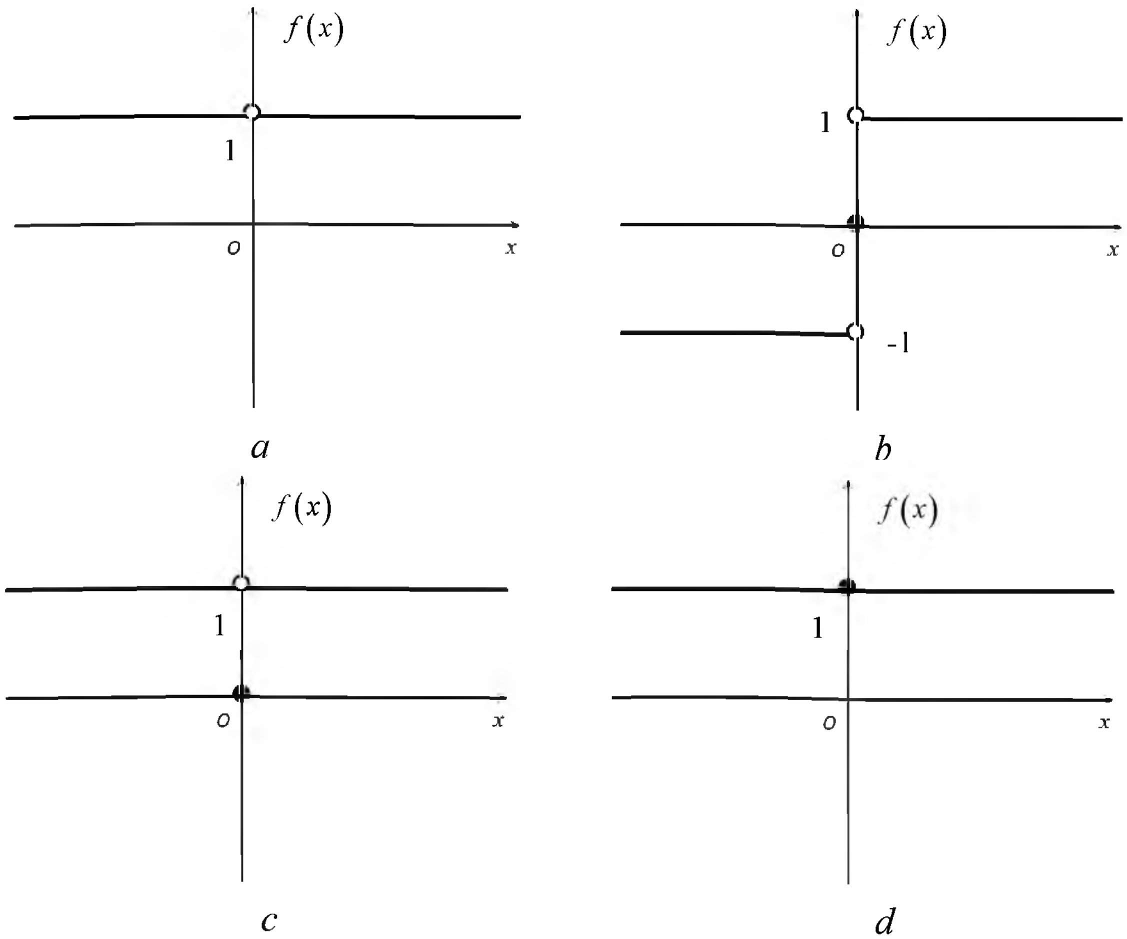
$$f(x_0) = f\left(\lim_{x \to x_0} x\right) = \lim_{x \to x_0} f(x).$$

In other words the limit sign can be replaced with the function symbol f for a continuous function.

Example 2.28. Examine $f(x)$ for continuity at $x = 0$, if

$$\text{a)}\ f(x) = \frac{x}{x};\quad \text{b)}\ f(x) = \operatorname{sgn}(x);\quad \text{c)}\ f(x) = |\operatorname{sgn}(x)|;\quad \text{d)}\ f(x) = \begin{cases} \dfrac{x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

□ Plot graphs of the given functions first.



Pic. 2.10

a) $f(x)=\frac{x}{x}$ is not defined at $x=0$. Thus, $f(x)$ is not continuous at $x=0$ because of failure of condition a (see the definition given above).

b) Condition a is met for $f(x)=\text{sgn}(x)$ as $\text{sgn}(0)=0$. But condition b is failed. As it's known if $\lim\limits_{x\to x_0-0} f(x)=\lim\limits_{x\to x_0+0} f(x)$ then $\lim\limits_{x\to x_0} f(x)$ exists and vice versa. In the case $\lim\limits_{x\to -0} f(x)=-1\neq \lim\limits_{x\to +0} f(x)=1$, that implies nonexistence of $\lim\limits_{x\to 0} f(x)$. Hence, $f(x)$ is not continuous at $x=0$.

c) As it's shown in pic. $f(0)$ exists ($f(0)=0$) and $\lim\limits_{x\to 0} f(x)$ exists ($\lim\limits_{x\to -0} f(x)=\lim\limits_{x\to +0} f(x)=1$). But $\lim\limits_{x\to 0} f(x)=1\neq f(0)=0$ or condition c isn't met. So $f(x)$ is not continuous at $x=0$.

d) $f(x)$ is continuous at $x=0$ because $f(0)$ exists ($f(0)=1$), $\lim\limits_{x\to 0} f(x)$ exists ($\lim\limits_{x\to -0} f(x)=\lim\limits_{x\to +0} f(x)=1$) and $\lim\limits_{x\to 0} f(x)=f(0)=1$. ■

Notice, that drawing the graph of a continuous function can be carried out without any breaks.

The (ε,δ) - definition of continuity can be stated as follows:

Def.: $f(x)$ is said to be ***continuous*** at x_0 if for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $|f(x)-f(x_0)|<\varepsilon$ whenever $|x-x_0|<\delta$.

Let $x-x_0=\Delta x$ (x is a point in a δ - neighborhood of x_0 where $f(x)$ is defined). Then $\lim\limits_{x\to x_0} f(x)=f(x_0)$ can be rewritten as

$$\lim_{x\to x_0} f(x_0+\Delta x)=f(x_0).$$

According to the definition of the limit $|f(x_0+\Delta x)-f(x_0)|<\varepsilon$. The expression $f(x_0+\Delta x)-f(x_0)$ is Δf called the ***increment*** of f. Further denote $f(x_0+\Delta x)-f(x_0)$ as $\alpha(x)$, i.e. $f(x_0+\Delta x)-f(x_0)=\alpha(x)$. So $|\alpha(x)|<\varepsilon$ that means $\alpha(x)$ is infinitesimal as $x\to x_0$ or $\lim\limits_{x\to x_0}\alpha(x)=0$. Hence, since $\Delta x\to 0$ as $x\to x_0$ we get one more form of the definition of continuity:

$$\lim_{x\to x_0}\Delta f=0.$$

Example 2.29. Verify continuity of $f(x)=x^2$ at any real point x.

□ Using $\lim\limits_{x\to x_0}\Delta f=0$, we have

$$\lim_{\Delta x\to 0}\Delta f=\lim_{\Delta x\to 0}\left(f(x+\Delta x)-f(x)\right)=\lim_{\Delta x\to 0}\left((x+\Delta x)^2-x^2\right)=$$

$$= \lim_{\Delta x \to 0} \left(x^2 - 2x\Delta x + (\Delta x)^2 - x^2 \right) = 0 .$$

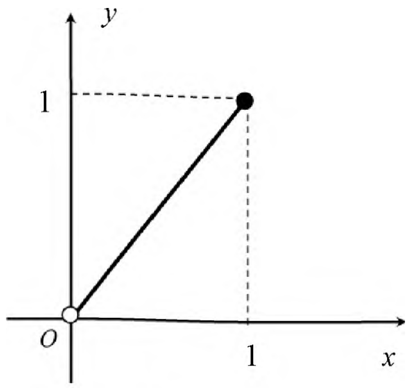
Thus, $f(x) = x^2$ is continuous at any point x .■

Def.: $f(x)$ is called ***right continuous*** at x_0 if $\lim\limits_{x \to x_0 + 0} f(x) = f(x_0 + 0)$.

Def.: $f(x)$ is called ***left continuous*** at x_0 if $\lim\limits_{x \to x_0 - 0} f(x) = f(x_0 - 0)$.

Remark 2.7

Consider $f(x) = x, x \in (0,1]$ (see pic. 2.11). $f(x)$ is right continuous at $x = 1$, but $f(x)$ is not continuous at $x = 0$ ($f(0)$ doesn't exist).



Pic. 2.11

Def.: $f(x)$ is called ***continuous on*** X if $f(x)$ is continuous at every point $x \in X$.

Properties of continuous functions

Let $f(x)$ and $g(x)$ be defined in some neighborhood of x_0 including x_0 itself and continuous at x_0 .

1. Moreover, if $f(x)$ is continuous in the considered neighborhood of x_0 and $f(x_0) \neq 0$ there exists a neighborhood of x_0 where $f(x) \neq 0$ and $f(x)$ keeps its sign (the sign of $f(x_0)$).

2. $f(x) \pm g(x), f(x) \cdot g(x), \dfrac{f(x)}{g(x)} \left(g(x) \neq 0 \right)$ are continuous at x_0 .

3. A composite function $f(g(x))$ is continuous at x_0 and

$$\lim_{x \to x_0} f(g(x)) = f\left(\lim_{x \to x_0} g(x) \right) = f(g(x_0)) .$$

4. All basic elementary functions are continuous on their domain.

Def.: x_0 is said to be a ***point of discontinuity*** if $f(x)$ is not continuous at x_0.

Classification of points of discontinuity

Let $A = f(x_0 - 0) = \lim\limits_{x \to x_0 - 0} f(x)$ and $B = f(x_0 + 0) = \lim\limits_{x \to x_0 + 0} f(x)$.

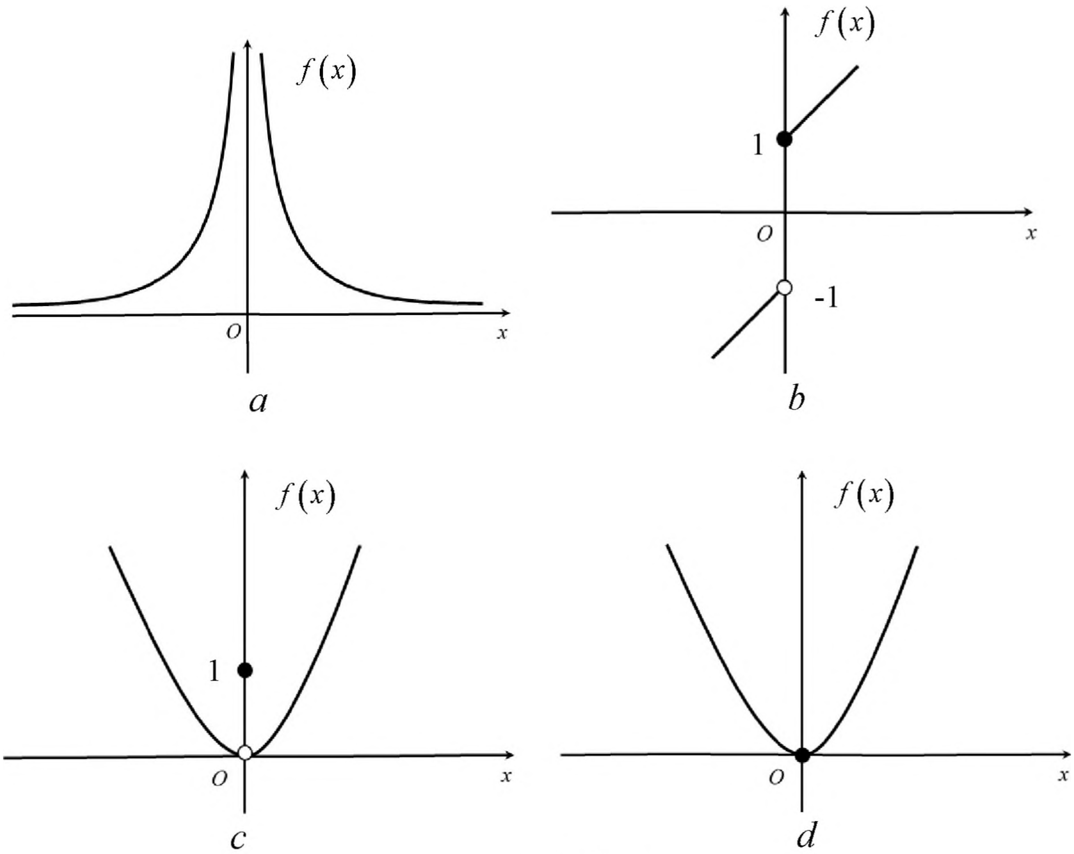
1. If A, B exist, $A, B = \text{const}$ (they might take different values), but $A \neq B$ then $f(x)$ is said to have a ***jump discontinuity*** and x_0 is a point of discontinuity of the first kind.
2. If A, B exist, $A, B = \text{const}$ and $A = B$ then $f(x)$ is said to have a ***removable discontinuity*** and x_0 is a point of removable discontinuity.
3. In all other cases $f(x)$ has an ***essential discontinuity*** and x_0 is a point of discontinuity of the second kind.

Example 2.30. Examine $f(x)$ for continuity, if

a. $f(x) = \dfrac{1}{x^2}$;   b. $f(x) = \begin{cases} x+1, & \text{if } x \geq 0, \\ x-1, & \text{if } x < 0; \end{cases}$

c. $f(x) = \begin{cases} x^2, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0; \end{cases}$ d. $f(x) = x^2$.

□ Plot graphs of the given functions first.



Pic. 2.12

a. $f(x)$ is defined at every real point x except $x=0$. So $f(x)$ is not continuous at $x=0$ (condition a is failed). It's obvious that $\lim\limits_{x\to-0} f(x)=\lim\limits_{x\to+0} f(x)=+\infty$. Thus $f(x)$ has an essential discontinuity at $x=0$ and $x=0$ is a point of discontinuity of the second kind.

b. $f(x)$ is continuous on $(-\infty,0)\cup(0,+\infty)$ as a linear function. The only point that interests us is $x=0$. Condition a is met for $f(x)$ ($f(0)$ exists, $f(0)=1$) while condition b is invalid ($\lim\limits_{x\to-0} f(x)=-1\neq\lim\limits_{x\to+0} f(x)=1$). So $f(x)$ is not continuous at $x=0$. Since both one-sided limits exist and are constant, $f(x)$ has a jump discontinuity at $x=0$ and $x=0$ is a point of discontinuity of the first kind.

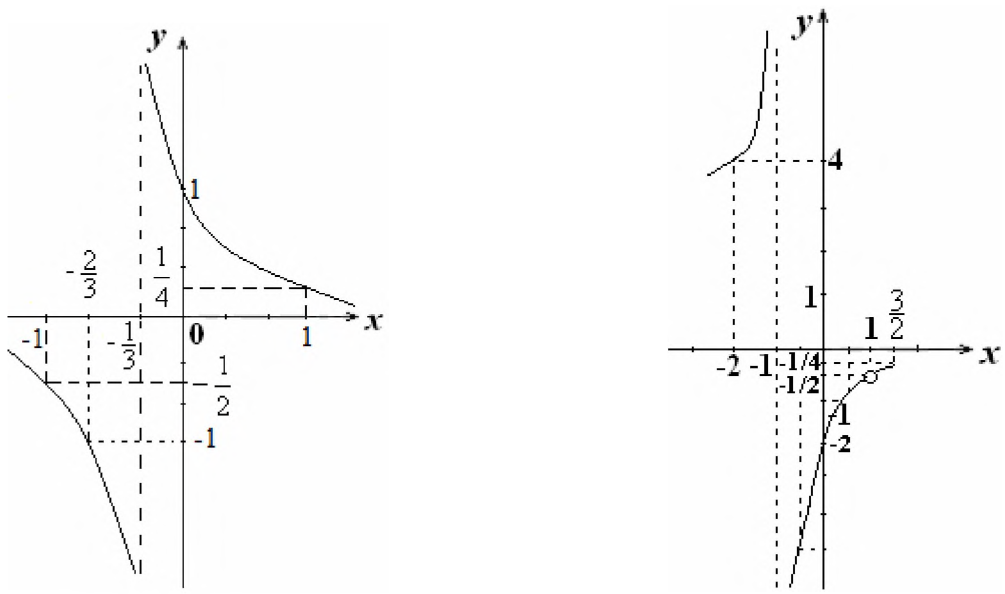
c. By analogy with case b $x=0$ is the only point of discontinuity of $f(x)$. Condition a and b are met for $f(x)$ ($f(0)$ exists, $f(0)=1$; $\lim\limits_{x\to-0} f(x)=\lim\limits_{x\to+0} f(x)=0$). However condition c is failed because $\lim\limits_{x\to-0} f(x)=\lim\limits_{x\to+0} f(x)=0\neq f(0)=1$. Hence, $f(x)$ has a removable discontinuity at $x=0$ and $x=0$ is a point of removable discontinuity.

d. $f(x)$ is continuous for all real x even at $x=0$ because $f(0)$ exists, $f(0)=0$; $\lim\limits_{x\to-0} f(x)=\lim\limits_{x\to+0} f(x)=0=f(0)$.■

Example 2.31. Examine $f(x)$ for continuity at $x=-\frac{1}{3}$, if $f(x)=\frac{1}{3x+1}$.

□ The elementary function $f(x)=\frac{1}{3x+1}$ is continuous for all real x except a point where the denominator $3x+1=0$, i.e. except $x=-\frac{1}{3}$. So $x=-\frac{1}{3}$ is a point od discontinuity of $f(x)$. Let's classify the point. For this reason we should evaluate one- sided limits at $x=-\frac{1}{3}$: $\lim\limits_{x\to-\frac{1}{3}-0}\frac{1}{3x+1}=-\infty$; $\lim\limits_{x\to-\frac{1}{3}+0}\frac{1}{3x+1}=+\infty$.

It means $f(x)$ has an essential discontinuity at $x=-\frac{1}{3}$ and $x=-\frac{1}{3}$ is a point of discontinuity of the second kind. Behavior of $f(x)$ near $x=-\frac{1}{3}$ is illustrated below (pic. 2.13, *a*). ■



a *b*

Pic. 2.13

Example 2.32. Examine $f(x)$ for continuity, if $f(x)=\dfrac{x^2-3x+2}{x^2-1}$.

□ $f(x)$ can be considered as a ratio of two polynomials, that are continuous everywhere. So $f(x)$ is continuous for all real x except points where $x^2-1=0$, i.e. $x=\pm1$ (see property 2 for continuous functions). Thus $x=\pm1$ are points of discontinuity of $f(x)$. In order to classify them, one-sided limits at $x=\pm1$ should be calculated.

Let $x=-1$

$$\lim_{x\to-1+0} f(x)=\lim_{x\to-1+0}\frac{x^2-3x+2}{x^2-1}=\lim_{x\to-1+0}\frac{(x-1)(x-2)}{(x-1)(x+1)}=\lim_{x\to-1+0}\frac{x-2}{x+1}=-\infty,$$

$$\lim_{x\to-1-0} f(x)=\lim_{x\to-1-0}\frac{x^2-3x+2}{x^2-1}=\lim_{x\to-1-0}\frac{(x-1)(x-2)}{(x-1)(x+1)}=\lim_{x\to-1-0}\frac{x-2}{x+1}=+\infty.$$

Now let $x=1$

$$\lim_{x\to1-0} f(x)=\lim_{x\to1-0}\frac{(x-1)(x-2)}{(x-1)(x+1)}=\left[\frac{0}{0}\right]=\lim_{x\to1-0}\frac{x-2}{x+1}=-\frac{1}{2},$$

$$\lim_{x\to1+0} f(x)=\lim_{x\to1+0}\frac{(x-1)(x-2)}{(x-1)(x+1)}=\left[\frac{0}{0}\right]=\lim_{x\to1+0}\frac{x-2}{x+1}=-\frac{1}{2}.$$

According to the obtained results we conclude that $f(x)$ has an essential discontinuity at $x=-1$, $x=-1$ is a point of discontinuity of the second kind and $f(x)$ has a removable discontinuity at $x=1$ and $x=1$ is a point of removable discontinuity. The graph of $f(x)$ is given above (pic. 2.13, *b*). ■

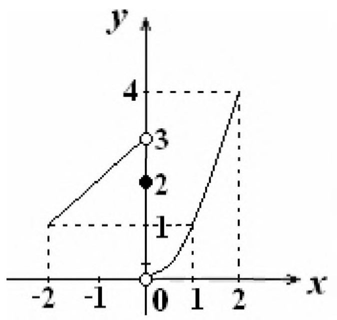
Example 2.33. Examine $f(x)$ for continuity, if

$$f(x)=\begin{cases} x+3, \text{ if } x\in[-2;0), \\ 2 \quad , \text{ if } x=0, \\ x^2 \quad , \text{ if } x\in(0,2]. \end{cases}$$

□ $f(x)$ is referred to as a piecewise function. As it follows from its analytical representation, $f(x)$ is continuous on $[-2,0)\cup(0,2]$ as elementary function considered on its domain. We have interest in analyzing behavior of $f(x)$ at $x=0$. Calculate the one-sided limits at $x=0$:

$$\lim_{x\to-0} f(x)=\lim_{x\to-0}(x+3)=3; \ \lim_{x\to+0} f(x)=\lim_{x\to+0} x^2=0.$$

Summarizing the above $f(x)$ has a jump discontinuity and $x=0$ is a point of discontinuity of the first kind ($\lim_{x\to-0} f(x)$, $\lim_{x\to+0} f(x)$ exist, but $\lim_{x\to-0} f(x)=3\neq\lim_{x\to+0} f(x)=0$). The graph of $f(x)$ is drawn below. ■



Pic. 2.14

Properties of continuous functions on closed intervals

Theorem 2.11 (The Intermediate Value Theorem, IVT)

If $f(x)$ is continuous on $[a,b]$ and $f(a)=A$, $f(b)=B$, then for any C: $A<C<B$ there exists at least one c such that $f(c)=C$.

Corollary

If $f(x)$ is continuous on $[a,b]$ and attains values with opposite signs at the endpoints a and b, then $f(x)$ takes zero value at least one point within $[a,b]$.

Theorem 2.12 (the Weierstrass Extreme Value Theorem)

If $f(x)$ is continuous on $[a,b]$, then $f(x)$ attains its extreme values on it.

Example 2.34. Show that $f(x)=x^3+x$ takes the value 9 for some x in $[1,2]$.

□ $f(x)=x^3+x$ is continuous on $[1,2]$ as an elementary function (a polynomial) considered on a subset of its domain. Moreover, $f(1)=1^3+1=2$ and $f(2)=2^3+2=10$. Since 9 is an intermediate value between 2 and 10, the IVT says that there is a point x such that $f(x)=9$. ■

Exercises

1. Find points of discontinuity of $f(x)$ and classify them if any

a) $f(x)=\dfrac{\sin x}{x}$; b) $f(x)=\dfrac{1}{\cos^2 x}$; c) $f(x)=\arctan\dfrac{1}{x-5}$;

d) $f(x)=\dfrac{1}{2-2^{1/x}}$; e) $f(x)=\sin\dfrac{1}{x}$.

2. Using the definition prove that the following functions are continuous at every real point $x\in\mathbb{R}$:

a) $f(x)=x^3$;

b) $f(x)=\sin x$;

c) $f(x)=x^2-5x+2$.

3. Examine $f(x)$ for continuity on intervals $[0,2],[-3,1],[4,5]$:

a) $f(x)=\dfrac{1}{x^4-1}$;

b) $f(x)=\ln\dfrac{x+4}{x-5}$;

c) $f(x)=\dfrac{1}{x^2+2x-3}$.

CHAPTER 3. DIFFERENTIAL CALCULUS

3.1. DERIVATIVES AND THEIR APPLICATIONS

Let $y = f(x)$ be a function defined in a neighborhood $U(x_0)$ of a point x_0. Increase x by Δx so that $x = x_0 + \Delta x$ remains in the same neighborhood $U(x_0)$. Δx is called the ***increment of the independent variable*** x. Then $\Delta f = f(x) - f(x_0)$ is the ***increment of the function*** $y = f(x)$, caused by changing x. Δf can be written as $\Delta f = f(x_0 + \Delta x) - f(x_0)$. Notice, Δf depends on both x_0 and Δx : $\Delta f(x_0, \Delta x)$.

Def.: A ***derivative*** $f'(x_0)$ of the function $y = f(x)$ at x_0 is called the limit of the ratio of Δf to Δx $(\Delta x \neq 0)$ as $\Delta x \to 0$:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}, \tag{3.1}$$

if it (the limit) exists.

Other notations for the derivative such as $y'(x_0)$, $\frac{dy}{dx}(x_0)$, $\frac{df}{dx}(x_0)$, $\frac{d}{dx} f(x_0)$, $\left.\frac{df}{dx}\right|_{x=x_0}$, $f'_x(x_0)$, $y'_x(x_0)$ are also used. The notation $\frac{df}{dx}, \frac{dy}{dx}$ involving differentials is referred to as the ***Leibniz's notation.*** The notation f', y' is ***Lagrange's notation***.

Remark 3.1

If the limit value in (3.1) is equal to a finite constant, i.e. $\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \text{const} < \infty$, then the derivative is called a ***finite derivative***. Otherwise the derivative is called an ***infinite derivative***. If the limit in (3.1) doesn't exist then the derivative $f'(x_0)$ doesn't exist as well.

Def.: A ***right-hand derivative*** $f'_+(x_0)$ of the function $y = f(x)$ at x_0 is

$$f'_+(x_0) = \lim_{\Delta x \to 0+0} \frac{\Delta f}{\Delta x}.$$

Def.: A ***left-hand derivative*** $f'_-(x_0)$ of the function $y = f(x)$ at x_0 is

$$f'_-(x_0) = \lim_{\Delta x \to 0+0} \frac{\Delta f}{\Delta x}.$$

Remark 3.2

At endpoints of a closed interval it can be defined one-sided derivatives only.

Theorem 3.1 (the necessary and the sufficient conditions for existence of a derivative at a point).

A function $f(x)$ *has a derivative* $f'(x_0)$ *at* x_0 *if and only if one-sided derivatives* $f'_+(x_0)$ *and* $f'_-(x_0)$ *exist. And* $f'(x_0)=f'_+(x_0)=f'_-(x_0)$.

Theorem 3.2 (the necessary condition for existence of a finite derivative at a point).

If a function $f(x)$ *has a finite derivative* $f'(x_0)$ *at* x_0 *, the function* $f(x)$ *is continuous at* x_0 *.*

Corollary

If a function $f(x)$ *is not continuous at* x_0 *, the function* $f(x)$ *has no finite derivative at* x_0 *.*

3.2. DIFFERENTIABILITY OF A FUNCTION AND DIFFERENTIAL

Def.: A function $f(x)$ is called ***differentiable*** at x_0 , if Δf can be expressed as

$$\Delta f = A\cdot\Delta x + o(\Delta x), \tag{3.2}$$

in a neighborhood $U(x_0)$, where A is a finite constant, not depending on Δx , $o(\Delta x)$ is an infinitesimal of higher order than Δx as $\Delta x\to 0$: $\lim\limits_{\Delta x\to 0} o(\Delta x)=0$ and $\lim\limits_{\Delta x\to 0}\dfrac{o(\Delta x)}{\Delta x}=0$.

Def.: The first term in (3.2) which is a linear function of Δx is called the ***differential*** of $f(x)$. The differential is denoted as df. Other notations for differential as $dy(x_0)$, $dy(x_0,\Delta x)$, $df(x_0)$. are also used.

Thus (3.2) can be rewritten as $\Delta f = df + o(\Delta x)$, $\Delta x\to 0$.

Theorem 3.3. (the necessary and the sufficient conditions for differentiability of a function)

A function $f(x)$ *is differentiable at* x_0 *if and only if a finite derivative* $f'(x_0)$ *exists, and*

$$df(x_0)=f'(x_0)\cdot dx \text{ or } df=f'\cdot dx , \tag{3.3}$$

where $dx=\Delta x$.

Remark 3.3

1. $dx=\Delta x$ is met only if x is an independent variable.

2. The given above theorem is still true even if x_0 is one of the endpoints of a closed interval. But, in this case, $f'(x_0)$ should be replaced with a one-sided derivative.

Def.: The process of calculating derivatives and differentials is called ***differentiation***.

Def.: A function $f(x)$ is called ***differentiable*** on an interval X if the function $f(x)$ is differentiable at each point in X.

Example 3.1. Verify that the function $f(x)=x$ is differentiable for all $x\in\mathbb{R}$.

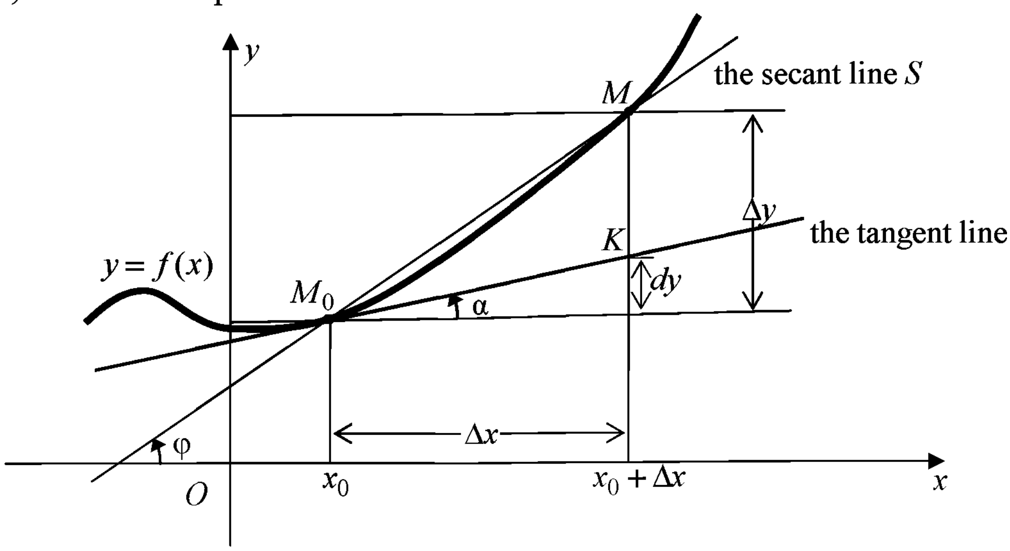
□ According to (3.1) for some point $x\in\mathbb{R}$ we have

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=$$
$$=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{x+\Delta x-x}{\Delta x}=1<\infty. \tag{3.4}$$

In (3.4) x can be any real number, so the derivative $f'(x)$ of the given function $y=x$ exists and it is finite for all $x\in\mathbb{R}$. Then according to theorem 3 the function $y=x$ is differentiable for all $x\in\mathbb{R}$. And (3.3) implies $df=dx$ for all $x\in\mathbb{R}$.■

Geometrical interpretation of a derivative

Consider the "finite derivative" case. Let $f(x)$ be a function, that is continuous in a neighborhood $U(x_0)$ of a point x_0 and has a finite derivative at x_0. Let $M_0(x_0,f(x_0))$ and $M(x_0+\Delta x,f(x_0+\Delta x))$ be two points that are located on the graph of the given function $f(x)$. Draw a secant line S passing through these points, as shown in pic. 1.



Pic. 3.1

Let φ be an angle between the secant line S and the positive direction of the x-axis. Draw a straight line that is parallel to the x-axis and passes through M_0. Denote the point of intersection of the straight line and the vertical line with x-intercept $x = x_0 + \Delta x$ by N. The angle φ is equal to $\angle MM_0N$. Obviously, if $f(x)$ is an increasing function, then $\Delta f = MN$ and $\Delta x = M_0N$.

As $\Delta x \to 0$ moving along the given curve the point M gets closer and closer to M_0. The secant line $S = M_0M$ tends to occupy the position of the straight line M_0K. The straight line M_0K is called a ***tangent line*** to the given curve at M_0.

The slope of the secant line S denoted by k_S, is equal to $\tan(\angle MM_0N)$: $k_S = \tan\varphi$, where $\tan\varphi = \dfrac{\Delta f}{\Delta x}$. The slope of the tangent line M_0K denoted by k, can be defined as follows:

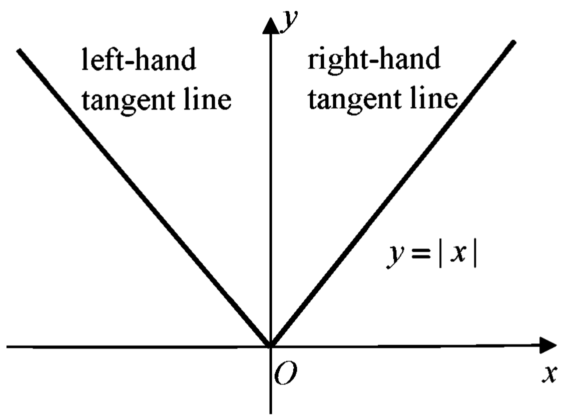
$$k = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \Rightarrow k = f'(x_0) = \tan\alpha.$$

Hence, ***the value of a derivative*** $f'(x_0)$ ***is equal to the slope of a tangent line.***

The point $K(x_K; y_K)$ is a point of intersection of the tangent line M_0K and the vertical line $x = x_0 + \Delta x$, so $x_K = x_0 + \Delta x$ and $y_K = f(x_0) + KN$, where $KN = M_0N \cdot \tan\alpha = \Delta x \cdot f'(x_0) = f'(x_0)dx$. By (3.3) $KN = df(x_0)$. Thus, the ***value of the differential*** $df(x_0)$ ***is equal to the increment of the ordinate of a tangent line to a curve*** $y = f(x)$ ***at*** x_0.

Let $f(x)$ be continuous at x_0. Assume, that $f(x)$ is not differentiable, but has finite one-sided derivatives $f'_-(x_0)$ and $f'_+(x_0)$ at x_0, that are not equal $f'_-(x_0) \neq f'_+(x_0)$. So, only one-sided tangent lines, the left-hand tangent line and the right-hand tangent line, can be passed through the point $(x_0, f(x_0))$. The slopes of these one-sided tangent lines are equal to $f'_-(x_0)$ and $f'_+(x_0)$ respectively.

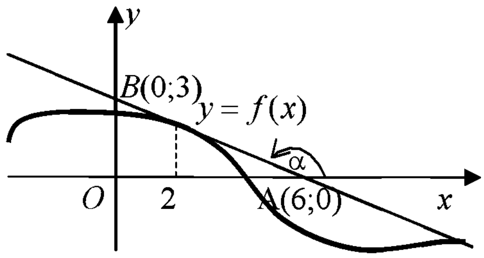
Example 3.2. Consider the function $y = |x|$. This function doesn't have a derivative at the point $x = 0$. But the given function has one-sided derivatives $y'_-(0) = -1$ and $y'_+(0) = -1$ at this point. So the half-line $y = x, x \geq 0$ is both a part of the graph of the given function $y = |x|$ and the right-hand tangent line. By analogy the half-line $y = -x, x \leq 0$ is the left-hand tangent line (see pic. 3.2).



Pic. 3.2

Example 3.3. Calculate $f'(2)$ if the tangent line to the given curve $y = f(x)$ at the point $(2; f(2))$ intersects the x-axis at the point $A(6;0)$ and the y -axis at the point $B(3; 0)$ as shown in pic. 3.3.

□ The slope k of the tangent line at the point $(2; f(2))$ is equal to $f'(2)$. The tangent line is a straight line, passed through the points A and B. Then $k = \frac{y_A - y_B}{x_A - x_B} = \frac{0-3}{6-0} = -\frac{3}{6} = -0,5$. ■



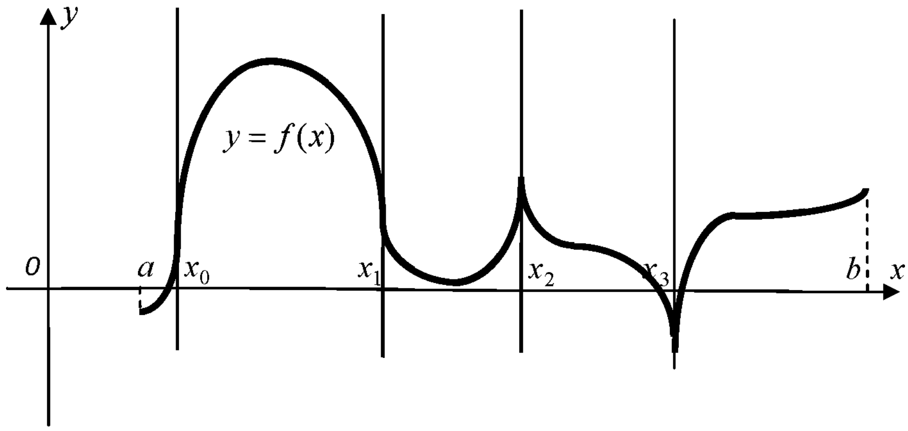
Pic. 3.3

Consider the "infinite derivative" case. Let $f(x)$ be continuous on a closed interval $[a, b]$ and have infinite derivatives at points $x_i \in [a, b]$, $i = 0, ..., 3$: $f'(x_i) = \infty$. Then, vertical lines $x = x_i$ are tangent lines to the given curve $y = f(x)$ at points $(x_i, f(x_i))$, $i = 0,...,3$.

In pic. 3.4 there are shown vertical tangent lines at points $(x_i, f(x_i))$, $i = 0,...,3$ for the following values of one-sided derivatives:

a) $f'_-(x_0) = f'_+(x_0) = +\infty$; b) $f'_-(x_0) = f'_+(x_0) = -\infty$;

c) $f'_-(x_0) = +\infty$; $f'_+(x_0) = -\infty$; d) $f'_-(x_0) = -\infty$; $f'_+(x_0) = +\infty$.



Pic. 3.4

Differentiation rules

Any elementary function $y = f(x), x \in X$ has a derivative $f'(x)$ at each point $x \in X$. Note, $f'(x)$ is an elementary function as well. The table of derivatives of a few basic elementary functions is given below.

The table of derivatives

$$(C)' = 0,\ C = \text{const}$$

$$\left(x^n\right)' = n \cdot x^{n-1}$$

$$\left(a^x\right)' = a^x \cdot \ln a,\ a > 0$$

$$\left(e^x\right)' = e^x$$

$$\left(\log_a x\right)' = \frac{1}{x \cdot \ln a}, a > 0\ a \neq 1;$$

$$\left(\ln x\right)' = \frac{1}{x}$$

$$\left(\sin x\right)' = \cos x$$

$$\left(\cos x\right)' = -\sin x$$

$$\left(\tan x\right)' = \frac{1}{\cos^2 x}$$

$$\left(\cot x\right)' = -\frac{1}{\sin^2 x}$$

$$\left(\sin^{-1} x\right)' = \left(\arcsin x\right)' = \frac{1}{\sqrt{1-x^2}}$$

$$\left(\cos^{-1} x\right)' = \left(\arccos x\right)' = -\frac{1}{\sqrt{1-x^2}}$$

$$\left(\tan^{-1} x\right)' = \left(\arctan x\right)' = \frac{1}{1+x^2}$$

$$\left(\cot^{-1} x\right)' = \left(\text{arccot}\, x\right)' = -\frac{1}{1+x^2}$$

$$\left(\text{sh}x\right)' = \text{ch}x$$

$$\left(\text{ch}x\right)' = \text{sh}x$$

$$\left(\text{th}x\right)' = \frac{1}{\text{ch}^2 x}$$

$$\left(\text{cth}x\right)' = -\frac{1}{\text{sh}^2 x}$$

Theorem 3.4 (The Sum Rule, The Difference Rule, The Product Rule, The Constant Multiple Rule, The Quotient Rule)

If functions $u = u(x)$ and $v = v(x)$, defined on a neighborhood of a point x_0, are differentiable at x_0, then the sum, the difference, the product and the quotient of these functions will be differentiable at this point x_0 too. The corresponding rules are listed below.

For derivatives: For differentials:

The Sum Rule

$$(u+v)' = u' + v' \qquad d(u+v) = du + dv$$

The Difference Rule

$$(u-v)' = u' - v' \qquad d(u-v) = du - dv$$

The Product Rule

$(u \cdot v)' = u'v + uv'$ $\qquad$ $d(u \cdot v) = du \cdot v + u \cdot d\,v$

The Constant Multiple Rule

$(Cu)' = C \cdot u',\ C = \text{const}$ $\qquad$ $d(Cu) = C \cdot du,\ C = \text{const}$

The Quotient Rule

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2},\ v(x_0) \neq 0.$ $\qquad$ $d\left(\frac{u}{v}\right) = \frac{du \cdot v - u \cdot dv}{v^2}, v(x_0) \neq 0.$

Physical interpretation of a derivative

Let $S(t)$ be a distance between a position of an object at the moment of time t and a position at $t + \Delta t$. It is assumed, that the object is moving rectilinearly. The ***average velocity*** $\hat{v}(t)$ during the time interval $[t, t + \Delta t]$ is defined by

$$\hat{v}(t) = \frac{\Delta S}{\Delta t},$$

where $\Delta S = S(t + \Delta t) - S(t)$. The velocity $v(t)$ or the ***instantaneous velocity*** of the object at the moment of time t can be defined as the limit of the average velocity as $\Delta t \to 0$, i.e.

$$v(t) = \lim_{\Delta t \to 0} \hat{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}.$$

So accordingly to the given above definition the velocity $v(t)$ at the moment of time t is equal to the derivative of the distance $S(t)$ with respect to time t, i.e.

$$v(t) = S'(t).$$

As we know by (3.3) the deferential $dS(t)$ can be written as

$$dS(t) = S'(t)\Delta t.$$

Assume that the object is moving rectilinearly with the instant velocity $S'(t)$. Then $dS(t)$ is a distance travelled by the object during the time interval $[t, t + \Delta t]$.

Let $f(x)$ is a function defined in the interval $[x, x + \Delta x]$. The ***mean rate of change*** of the function $f(x)$ is

$$\hat{v}(x) = \frac{\Delta f}{\Delta x}.$$

The ***instantaneous rate of change*** can be expressed by

$$v(x)=\lim_{\Delta x\to 0}\hat{v}(x)=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}.$$

Obviously, according to the definition $v(x)=f'(x)$.

By analogy, we can define acceleration $a(t)$ of the object as the derivative of the velocity $v(t)$ with respect to time t, i.e.

$$a(t)=v'(t).$$

Remark 3.4

If we need to find a derivative $f'(x)$ at a given point x_0, then we should

- find the derivative $f'(x)$ at an arbitrary point x, where x is in the domain of $f(x)$;
- substitute x_0 for x into the obtained result.

If we don't need to find specific value of the derivative, we just define the derivative at an arbitrary point.

Example 3.4. Find the derivative f' and the differential df, if $f=\sqrt[3]{x^2}$.

□ Using the table of derivatives (position 2), for $n=2/3$ we have

$$\left(\sqrt[3]{x^2}\right)'=\left(x^{2/3}\right)'=\frac{2}{3}\cdot x^{2/3-1}=\frac{2}{3}\cdot x^{-1/3}=\frac{2}{3}\cdot\frac{1}{x^{1/3}}=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}};\; d(\sqrt[3]{x^2})=\frac{2dx}{3\sqrt[3]{x^2}}.\;\blacksquare$$

Example 3.5. Find the derivative f' and the differential df, if $f=2^x$

□ By the table of derivatives (position 3), for $a=2$ we get

$$\left(2^x\right)'=2^x\cdot\ln 2.\ \text{Then, } d(2^x)=2^x\ln 2dx.\blacksquare$$

Example 3.6. Find the derivative f' and the differential df, if $f=\log_3 x$

□ Applying the table of derivatives (position 4), for $a=3$ we have $\left(\log_3 x\right)'=\dfrac{1}{x\cdot\ln 3}$. Consequently, $d\log_3 x=\dfrac{dx}{x\ln 3}.\blacksquare$

Example 3.7. Find $f'(x)$ if $f(x)=(x-1)e^x$.

□ Using the Product Rule (see theorem 3.4), $f'(x)=((x-1)e^x)'=(x-1)'\cdot e^x+(x-1)\cdot(e^x)'$. Then by the Difference Rule and the table of derivatives we can write $(x-1)'=x'-1'=1-0=1$; $(1)'=0$; $(e^x)'=e^x$. Hence, $f'(x)=1\cdot e^x+(x-1)\cdot e^x=e^x(1+x-1)=e^x\cdot x.\blacksquare$

Example 3.8. Find a velocity $v(t)$ at $t=10$, if the distance $S(t)=3t^2+4t-5$.

□ We know, $v(10)=S'(10)$. So by the Sum and Constant Multiple Rule (see theorem 3.4) and also the table of derivatives, we get

$$v(10)=v(t)\big|_{t=10}=S'(t)\big|_{t=10}=(3t^2+4t-5)'\big|_{t=10}=$$
$$=6t+4\big|_{t=10}=6\cdot 10+4=64. \quad ■$$

3.3. DIFFERENTIATION OF A COMPOSITE FUNCTION. CHAIN RULE

Theorem 3.5 (Chain rule)

If a function $u=g(x)$ is differentiable at a point x_0 and a function $y=f(u)$ is differentiable at a point u_0, $u_0=g(x_0)$, then a composite function $y=f(g(x))$ is differentiable at the point x_0. And its derivative is defined by

$$(f(g(x)))'\big|_{x=x_0}=(f(u))'\big|_{u=u_0}\cdot(g(x))'\big|_{x=x_0} \text{ or } f'_x=f'_u\cdot u'_x.$$

We can also write $\left.\dfrac{df}{dx}\right|_{x=x_0}=\left.\dfrac{df}{du}\right|_{u=u_0}\cdot\left.\dfrac{du}{dx}\right|_{x=x_0}$.

In other words, to differentiate a composite function we need:

- to identify an "outside function" and an "inside function";
- to differentiate the "outside function" leaving the "inside function" alone;
- to multiply the derivative of the "outside function" by the derivative of the "inside function".

Example 3.9. Use the Chain Rule to differentiate $f(x)=\sin\sqrt{x}$.

□ Let's represent $f(x)=\sin\sqrt{x}$ as $f(g(x))=\sin g(x)$, $g(x)=\sqrt{x}$. Therefore, the "outside function" is $f(u)=\sin u$, the "inside function" is $g(x)=\sqrt{x}$. Evaluating derivatives of each of them and using the Chain Rule, we get

$$(f(u))'=(\sin(u))'=\cos(u),(g(x))'=\left(\sqrt{x}\right)'=\frac{1}{2\sqrt{x}},$$

$$(f(g(x)))'=(f(u))'\cdot(g(x))'=\frac{\cos\sqrt{x}}{2\sqrt{x}}. \quad ■$$

Example 3.10. Find the differential of a composite function $y=f(g(x))$

□ By the definition $df=\left(f(g(x))\right)' dx$. According to the Chain Rule we can write $df=\left(f(u)\right)' \cdot \left(g(x)\right)' dx$. As $\left(g(x)\right)' dx = dg(x) = \left|u=g(x)\right| = du$ we have $df=\left(f(u)\right)' du$ or $df=f'_u \cdot du$. ■

Thus, *the form of the differential doesn't depend on whether the argument of a function is an independent variable or a function of another argument.*

3.4. DIFFERENTIATION OF AN IMPLICIT FUNCTION

Let $y=y(x)$ be an implicit function defined in *D*. It means, that $y(x)$ is the solution of the equation $F(x, y)=0$, which describes the functional relationship between an independent variable x and a dependent variable y. In other words, substituting $y(x)$ for y in $F(x, y)=0$ we obtain the identity $F(x, y(x))=0$, which is true for each $x \in D$.

Assume, the function $y(x)$ is differentiable on *D*. In this case, to find $y'(x)$ it is necessary to differentiate both sides of the equation $F(x, y(x))=0$ with respect to x and then to solve it for $y'(x)$. In addition, by (3.3) dy can be evaluated as $dy=y'(x)dx$.

Example 3.11. Assuming that the equation $x+y+\ln(y-x)=0$ determines a function $y(x)$ such that $y=y(x)$, find y' and dy.

□ Let's substitute $y(x)$ for y in the given equation $x+y+\ln(y-x)=0$. The obtained equation is $x+y(x)+\ln(y(x)-x)=0$. Then we differentiate both sides of the obtained equation with respect to x:

$$\begin{gathered} \left(x+y(x)+\ln(y(x)-x)\right)'_x=0, \\ 1+y(x)'+\frac{1}{y(x)-x}\cdot(y(x)'-1)=0. \end{gathered} \tag{3.5}$$

Further we will use y' instead of $y'(x)$. Now we need to solve (3.5) for y':

$$y'=\frac{x-y+1}{y-x+1}.$$

By (3.3) $dy=y'dx$, thus $dy=\dfrac{x-y+1}{y-x+1}dx$. ■

3.5. DIFFERENTIATION OF AN INVERSE FUNCTION

Theorem 3.6.

Let $y=f(x)$ be a strictly increasing (decreasing) continuous function on a neighborhood of a point x_0. Suppose, $f(x)$ is a differentiable function at the point x_0 and $f'(x_0)\neq 0$. Then the inverse function $x=f^{-1}(y)$ exists and is continuous and strictly increasing (decreasing) on a neighborhood of the point $y_0=f(x_0)$. Moreover, the inverse function $x=f^{-1}(y)$ is differentiable at the point $y_0=f(x_0)$ and its derivative at this point equals

$$(f^{-1}(y_0))'_y=\frac{1}{f'(x_0)} \text{ or } x'_y=\frac{1}{y'_x}.$$

Example 3.12. Find y' if $y=\sin^{-1}x$.

□ To find the derivative we use the above theorem . Notice, that functions $y=\sin^{-1}x,\ x\in[-1,1]$ and $x=\sin y,\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ are mutually inverse. By theorem 3.6 $x'_y=\frac{1}{y'_x}$. We can rewrite it as $y'_x=\frac{1}{x'_y}$. So for $x\in(-1;1)$ we get

$$(\sin^{-1}x)'=\frac{1}{(\sin y)'}=\frac{1}{\cos y}=\frac{1}{\left(+\sqrt{1-\sin^2 y}\right)}=\frac{1}{\sqrt{1-x^2}}.\ ■$$

3.6. LOGARITHMIC DERIVATIVE

Let $y=f(x)$ be a function, which has positive value at a point x_0. Moreover, $f(x)$ is differentiable at this point.

Def.: The derivative of the natural logarithmic function of $f(x)$ at the point x_0 is called ***the logarithmic derivative of the function*** $f(x)$ at this point. The logarithmic derivative is evaluated with the Chain rule:

$$(\ln f(x))'\big|_{x=x_0}=\frac{f'(x_0)}{f(x_0)}.$$

Hence, the derivative of a function is related to the logarithmic derivative of this function as follows $f'(x_0)=(\ln f(x))'\big|_{x=x_0}\cdot f(x_0)$.

Using the logarithmic derivative differentiating the listed below functions can be simplified:

a) $(u(x))^{v(x)}$,

b) $u_1(x)\cdot u_2(x)\ldots u_n(x)$,

c) $\dfrac{u_1(x)\cdot u_2(x)\ldots u_n(x)}{v_1(x)\cdot v_2(x)\ldots v_k(x)}$.

Algorithm of differentiating functions by use of the logarithmic derivative

1. Find the natural logarithmic function $\ln f(x)$ whose argument is a given positive valued function $f(x)$. Simplify the obtained result with logarithm properties.
2. Find the logarithmic derivative $(\ln f(x))'$.
3. Find the derivative $f'(x)$ by $f'(x) = (\ln f(x))' \cdot f(x)$.

Example 3.13. Find f' if $f = (x^2)^{\ln x}$.

□ We can represent the given function as $f = (x^2)^{\ln x} = (u(x))^{v(x)}$. So, according to the above list of functions we can find the derivative by the algorithm:

1. Find $\ln f$: $\ln f = \ln\left((x^2)^{\ln x}\right)$. Using the logarithm property: $\ln a^b = b \cdot \ln a$, we get $\ln f = \ln x \cdot \ln(x^2) \Leftrightarrow \ln f = \ln x \cdot 2\ln x \Leftrightarrow \ln f = 2(\ln x)^2$.
2. Find the logarithmic derivative $(\ln f)'$ with the Chain rule:
$$(\ln y)' = \left(2(\ln x)^2\right)' = 2 \cdot 2\ln x \cdot \frac{1}{x} = \frac{4\ln x}{x}.$$
3. Find the derivative $f'(x)$: $f' = (\ln f)' \cdot f \Rightarrow f' = \dfrac{4\ln x}{x} \cdot \left(x^2\right)^{\ln x}$. ■

3.7. DIFFERENTIATION OF A PARAMETRIC FUNCTION

Let a functional relationship between variables x and y be represented parametrically, i.e.

$$x = \varphi(t),\ y = \psi(t),\ t \in T.$$

Theorem 3.7.

If functions $\varphi(t)$ and $\psi(t)$ are differentiable at $t_0 \in T$ and $\varphi'(t_0) \neq 0$, then the function y as a function of x is differentiable at $x_0, x_0 = \varphi(t_0)$ and the derivative y'_x at x_0 is defined by

$$y'_x(x_0) = \frac{\psi'(t_0)}{\varphi'(t_0)} \quad or \quad y'_x = \frac{y'_t}{x'_t}.$$

Example 3.14. If $x = 1 + \cos t$, $y = \sin t$, find $\dfrac{dy}{dx}$ at $t = \dfrac{\pi}{4}$.

□ First we find the derivative as a function of t

$$\frac{dy}{dx} = y'_x = \frac{(\sin t)'}{(1 + \cos t)'} = \frac{\cos t}{(-\sin t)} = -\cot t.$$

Then, if $t = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -\cot t\Big|_{t=\frac{\pi}{4}} = -\cot\dfrac{\pi}{4} = -1$. ■

3.9. TANGENT AND NORMAL LINES TO A CURVE

The problem is to get equations of a tangent line and a normal line to a given curve Γ at a point $M(x_0,y_0)\in\Gamma$.

If a function, which represents the curve Γ, has the derivative at the point M, then both the tangent line and the normal line exist. Equations of these lines are uniquely determined by three parameters:

$$x_0,\ y_0 \text{ and } y_0',$$

where x_0, y_0 are coordinates of the point M which lines are passed through, y_0' is the value of the derivative $\frac{dy}{dx}$ at the point M. Notice, y_0' equals the slope of the tangent line.

Let's consider the following cases:

1. The curve is represented by $y=f(x)$. Then $y_0=f(x_0)$, $y_0'=f'(x_0)$.
2. The curve is represented by $x=\varphi(t)$, $y=\psi(t)$. Then

$$x_0=\varphi(t_0),\ y_0=\psi(t_0)\ ,\ y_0'=\frac{\psi'(t_0)}{\varphi'(t_0)}.$$

Remark 3.5

1. If we don't know x_0 or t_0, we need to find them from the problem formulation.
2. The parameters x_0, y_0 and y_0' can be tabulated as shown below

Table 3.1

x_0	y_0	y_0'

The form of tangent line's and normal line's equations is determined by the value of y_0'.

The following cases can be distinguished:

1. If $y_0'\neq 0$ and $y_0'\neq\infty$, then the equation of the tangent line is $y=y_0+y_0'\cdot(x-x_0)$ and the equation of the normal line is $y=y_0-\frac{1}{y_0'}(x-x_0)$.

2. If $y_0'=0$, then the equation of the tangent line is $y=y_0$ and the equation of the normal line is $x=x_0$.

3. If $y_0'=\infty$, then the equation of the tangent line is $x=x_0$, the equation of the normal line is $y=y_0$.

Remark 3.6

In the first case both the tangent line and the normal line are oblique lines, in the second case the tangent line is a horizontal line and the normal line is a vertical line, in the last case the tangent line is a vertical line and the normal line is a horizontal line.

Example 3.15. Find equations of the tangent line and the normal line to the curve $y=\sqrt[3]{x}$ at two points: a) $x_0=1$; b) $x_0=0$.

□ a) To get the desired equations we should determine three quantities: x_0, y_0 and y_0'. According to the problem formulation $x_0=1$. To find y_0 we should substitute x_0 in the given function: $y_0=f(x_0)=\sqrt[3]{x_0}=1$. Then differentiating $\sqrt[3]{x}$ with respect to x and substituting x_0 in the obtained result, we have

$$y_0'=f'(x_0)=\left(\sqrt[3]{x}\right)'\Big|_{x=1}=\left(x^{\frac{1}{3}}\right)'\Bigg|_{x=1}=\frac{1}{3}x^{-\frac{2}{3}}\Big|_{x=1}=\frac{1}{3}.$$

We deal with the first case, when $y_0'\neq 0$ and $y_0'\neq\infty$. Thus,

the equation of the tangent line is $y-1=\frac{1}{3}(x-1)$ or $y=\frac{1}{3}x+\frac{2}{3}$;

the equation of the normal line is $y-1=-3(x-1)$ or $y=-3x+4$.

b) For the second point $x_0=0$, so $y_0=f(x_0)=\sqrt[3]{0}=0$,

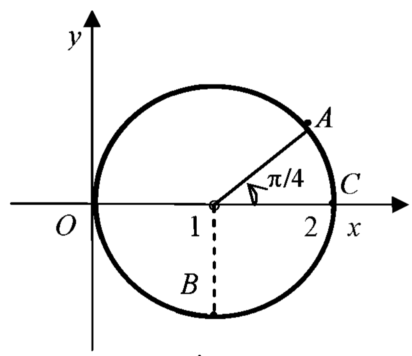
$y_0'=f'(0)=\frac{1}{3\sqrt[3]{x^2}}\Big|_{x=0}=\infty$. We deal with the third case, when $y_0'=\infty$. Therefore,

the equation of the tangent line is $x=0$; for the normal line, we have $y=0$. ■

Example 3.16. Find the tangent line and the normal line to the curve $x=1+\cos t,\ y=\sin t$ at points: a) $t_0=\frac{\pi}{4}$; b) $y_0=-1$; c) $x_0=2$.

□ Notice, the given curve $x=1+\cos t,\ y=\sin t$ is a circle with center at (1;0) and radius 1. Indeed, rewriting the equation $x=1+\cos t$ in the from $x-1=\cos t$ first and then raising the given parametric equations to the second power with adding the results together after, we get

$$+\begin{cases}(x-1)^2=\cos^2 t\\ y^2=\sin^2 t\end{cases}$$
$$\overline{(x-1)^2+y^2=1}$$



Pic. 3.5

The last equation is a canonical equation of a circle with center at (1;0) and radius 1 (see pic. 3.5).

We can use the result of example 3.14: $\frac{dy}{dx} = -\cot t$.

a) Evaluate x_0, y_0, y'_0: $t_0 = \frac{\pi}{4}$: $x_0 = 1 + \cos\frac{\pi}{4} = \frac{2+\sqrt{2}}{2}$, $y_0 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $y'_0 = -\cot\frac{\pi}{4} = -1$. The derivative $y'_0 \neq 0$, so we deal with the first case. Fill in the table 3.2

Table 3.2

x_0	y_0	y'_0
$\frac{2+\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1

Thus, the equation of the tangent line is $y = \frac{\sqrt{2}}{2} - 1 \cdot \left(x - \frac{2+\sqrt{2}}{2}\right)$ or $y = -x + 1 + \sqrt{2}$; the equation of the normal line is $y = \frac{\sqrt{2}}{2} + 1 \cdot \left(x - \frac{2+\sqrt{2}}{2}\right)$ or $y = x - 1$.

Coordinates of the point which both lines pass through are $\left(\frac{2+\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$.

b) In contrast to the previous case we have only *y*-coordinate of the point, not the value of variable t. So to find t it's necessary to solve the following equation $\sin t = -1 \Leftrightarrow t_0 = -\frac{\pi}{2} + 2\pi n,\ n \in Z$. Hence, $x_0 = 1 + \cos t_0 = 1$, $y'_0 = -\cot t_0 = 0$. $y'_0 = 0$, so we deal with the second case. Then the equation of the tangent line is $y = -1$; the equation of the normal line is $x = 1$. Both lines pass through the point $B(1; -1)$.

c) Now we have only *x*-coordinate of the point. By analogy with the above case solve the equation $2 = 1 + \cos t \Leftrightarrow \cos t = 1 \Leftrightarrow t_0 = 2\pi n,\ n \in Z$. Then $y_0 = \sin t_0 = \sin 2\pi n = 0$, $y'_0 = \cot t_0$. Notice, the value of y'_0 isn't defined. But we can determine it as $\lim_{t \to t_0} \cot t_0 = \infty$. We deal with the third case. The equation of the tangent line is $x = 2$; the equation of the normal line is $y = 0$. Both lines pass through the point $C(2;0)$. ■

Exercises

1. Find $f'(x)$, if

a) $f(x)=\sqrt{x}-\frac{3}{x}+\frac{9}{x^2}$;

b) $f(x)=x^3\log_2 x$;

c) $f(x)=\ln\sqrt[3]{\left(\frac{1-3x}{1+3x}\right)^2}$;

d) $f(x)=e^x\ln\sin x$;

e) $f(x)=\frac{\ln\cos x}{\cos x}$;

f) $f(x)=\ln\left(e^{2x}+1\right)-2\tan^{-1}e^x$.

2. Find $\frac{dy}{dx}$, if the function is represented parametrically:

a) $\begin{cases}x=2\cos t,\\ y=\sin t;\end{cases}$

b) $\begin{cases}x=2\tan t,\\ y=2\sin^2 t+\sin 2t;\end{cases}$

c) $\begin{cases}x=\sin t,\\ y=a^t;\end{cases}$

d) $\begin{cases}x=\sin t,\\ y=\cos 2t.\end{cases}$

3. Using the logarithmic derivative find $f'(x)$, if

a) $y=19^{x^{19}}x^{19}$;

b) $y=x^{e^{\cot x}}$;

c) $y=x^{3^x}\cdot 2^x$;

d) $y=x^{e^{\cos x}}$.

4. Find the equation of a tangent line to the curve $y=x^2-2x$ perpendicular to the line $3x+y-2=0$.

5. Find the equation of a tangent line to the curve $y^2=20x$ that makes the angle $\frac{\pi}{4}$ with the x - axis.

6. Find the equation of a tangent line to the curve $y=5x-x^2$ if the tangent line is parallel to the line passing through two points: $(1,11)$, $(-2,2)$.

In exercises 7,8 a student's number and the last numeral in a group number should be taken for m and n respectively.

7. Find derivatives y', if

a) $y=\frac{\ln(mx+n)}{x^2+n}$;

b) $y=x^{mx^2+n}$;

c) $\begin{cases}x=t^2+nt+1,\\ y=t^3+mt+1.\end{cases}$

8. Find the tangent line to the curve $y=x^3+nx+1$, if the tangent line is parallel to the line $y=2x+3$.

3.10. THE MAIN THEOREMS OF DIFFERENTIAL CALCULUS

Let a function $f(x)$ be defined in a neighborhood of a point x_0.

Theorem 3.8 (Fermat's Theorem).

If the function $f(x)$ has the largest value at the point x_0 and is differentiable at this point, then $f'(x_0)=0$.

This statement can be formulated in another way. But some terms must be defined first.

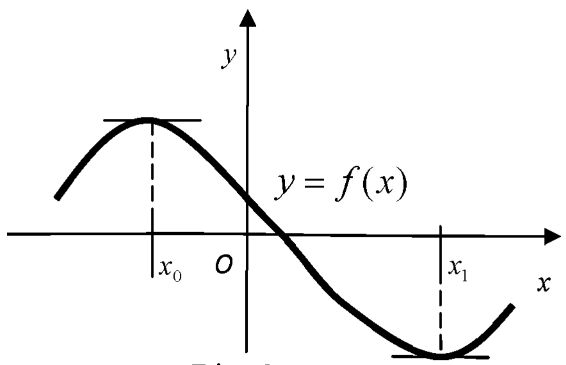
Def.: $f(x_0)$ is called a ***local maximum (minimum)*** of $f(x)$, if there exists a neighborhood of the point x_0 such that $f(x) \geq f(x_0)$ ($f(x) \leq f(x_0)$) for all x in this neighborhood.

Def.: A local minimum or a local maximum of the function $f(x)$ are called ***local extrema*** of $f(x)$.

Remark 3.7

We use the term "local" to underline that we deal with a small open interval such that $f(x)$ takes the largest (smallest) value. If $f(x)$ takes the largest or smallest value on some set X, then these values of $f(x)$ are called ***global extrema.***

According to the definitions above the Fermat's Theorem can be formulated as: *if the differentiable function $f(x)$ takes a local extrema at the point x_0, then $f'(x_0)=0$.*



Pic. 3.6

In pic. 3.6 the ***geometrical interpretation*** of this theorem is shown: the function has local maximum at the point x_0, and local minimum at the point x_1. We can see that tangent lines are parallel to x-axis at these points, because their slopes are zero: $k_{1,2}=f'(x_0)=f'(x_1)=\tan 0=0$ Thus,

a tangent line to the graph of the function $f(x)$ is parallel to the x-axis at points where the function has local extrema.

Theorem 3.9 (Rolle's Theorem).

Let a function $f(x)$

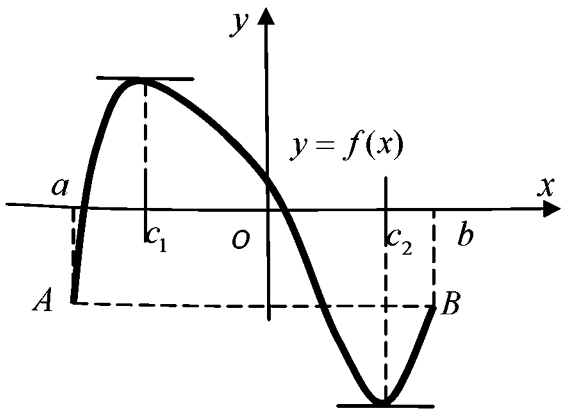
a) *be continuous on a closed interval $[a,b]$;*

b) *be differentiable on the open interval (a,b);*

c) *attains equal values at endpoints of $[a,b]$: $f(a)=f(b)$.*

Then, there exists at least one point c in (a,b) such that $f'(c)=0$.

The function $f(x)$ presented in pic. 3.7 satisfies all hypothesis of the Rolle's Theorem. The derivative $f'(x)$ is equal to zero at points c_1 and c_2. So the tangent lines at each of these points are parallel to x-axis and the chord AB, joining two points $(a, f(a))$ and $(b, f(b))$.



Pic. 3.7

Geometrical interpretation of the Rolle's Theorem: *if a function $f(x)$ satisfies all hypotheses of Rolle's Theorem, then there is at least one point on the graph of $f(x)$ such that a tangent line to the graph at this point is parallel to the x-axis.*

Theorem 3.10 (Cauchy's Mean Value Theorem, extended Mean Value Theorem).

Let functions $f(x)$ and $g(x)$

a) *be continuous in a closed interval $[a,b]$;*

b) *be differentiable on the open interval (a,b);*

c) $g'(x) \neq 0$ *for all* $x \in (a,b)$.

Then, there exists at least one c in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b.$$

Remark 3.8

The Cauchy's Mean Value Theorem is used to prove other main theorems. The Mean Value Theorem given below is the corollary of the Cauchy's Mean Value Theorem.

Theorem 3.11 (The Mean Value Theorem).

Let a function $f(x)$

a) *be continuous on a closed interval $[a,b]$;*

b) *be differentiable on the open interval (a,b).*

Then, there exists at least one c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}, \quad a < c < b.$

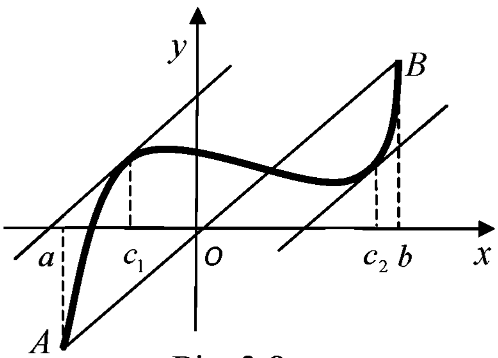
Following the Mean Value Theorem, $f(b) - f(a) = f'(c) \cdot (b-a)$.

Remark 3.9

If we denote $x_0 = a$, $x = x_0 + \Delta x = b$, $c = x_0 + \theta \cdot \Delta x$, where $\theta \in (0;1)$ and $\Delta y = f(x) - f(x_0)$, then the Mean Value formula can be rewritten as

$$\Delta y = f'(x_0 + \theta \cdot \Delta x) \cdot \Delta x .$$

Geometrical interpretation of the Mean Value Theorem: *if a function $f(x)$ satisfies all hypotheses of the Mean Value Theorem, then there is at least one point on the graph of $f(x)$ such that a tangent line to the graph at this point is parallel to the chord AB, joining two points $(a, f(a))$ and $(b, f(b))$.*



Pic. 3.8

Actually, in pic. 3.8 tangent lines at two points c_1 and c_2 are parallel to the chord AB. The quotient $\frac{f(b) - f(a)}{b - a}$ is equal to the slope of the secant line AB. We know, that slopes of parallel lines are equal. Hence, the Mean Value formula follows.

Example 3.17. Prove that the equation $f'(x) = 0$ has three different real roots, if $f(x) = x(x+1)(x+2)(x+3)$.

□ Since $f(x)$ is a polynomial it is continuous and differentiable on $\mathbb{R}$. Moreover, $f(x)$ takes the zero value at the points: $-3, -2, -1, 0$, that are roots of the given polynomial. Let's prove that $f'(x) = 0$ has a root in the interval $[-3;-2]$. $f(x)$ satisfies all hypothesis of the Rolle's Theorem on $[-3;-2]$. Indeed, $f(x)$ is continuous on $[-3;-2]$, differentiable on $(-3;-2)$ and $f(-3) = f(-2) = 0$. Consequently acoording to the Rolle's Theorem there is a point $c \in (-3;-2)$ such that $f'(c) = 0$. In a similar way we can prove that there are two more roots of the equation $f'(x) = 0$: one of them is in $(-2;-1)$ and the other one is in $(-1;0)$ ■

3.11. HIGHER-ORDER DERIVATIVES AND DIFFERENTIALS

Higher order derivatives

Let $y = f(x)$ be an explicit function, which has a finite derivative $y' = f'(x)$ at every point x in an interval X. If $f'(x)$ as a function of x has a derivative with respect to x which may be finite or infinite at every point x, $x \in X$, then this derivative is called the ***second derivative*** of $f(x)$ and denoted by

$$y'' = (f'(x))' \text{ or } y'' = f''(x).$$

In like manner, if $f''(x)$ as a function of x has a derivative for all $x \in X$ $(\forall x \in X)$, then this derivative is called the ***third derivative*** and denoted as

$$f'''(x) = (f''(x))'.$$

And so on, we use this way to obtain other higher order derivatives.

In general, if there exists a finite derivative of order $(n-1)$, i.e. $y^{(n-1)} = f^{(n-1)}(x)$, $\forall x \in X$, then the ***nth derivative*** can be defined as a derivative of the $(n-1)$ st derivative:

$$y^{(n)} = (f^{(n-1)}(x))' = f^{(n)}(x).$$

Obviously, if $n = 0$ we deal with the given function, i.e. $f^{(0)}(x) = f(x)$. If $n \geq 2$ ($n = 2; 3;...$), derivatives are called ***higher order derivatives***.

The following notations for the nth derivative: $f^{(n)}(x)$, $y^{(n)}$, $\dfrac{d^n y}{dx^n}$, $\dfrac{d^n f(x)}{dx^n}$ are used.

Remark 3.10

1. If the order of a derivative is known, for example, we need to find the second derivative, i.e. $n = 2$, then either Roman numerals without brackets or Arabic numerals within brackets can be used in notation for the required derivative. Using primes is also acceptable but for the second and the third derivatives only:

$$y'' = y^{\text{II}} = y^{(2)},$$
$$y''' = y^{\text{III}} = y^{(3)},$$
$$y^{\text{IV}} = y^{(4)}...$$

2. If the n-th derivative of $f(x)$ at x_0 exists, then its value at this point can be denoted in the following manner:

$$y^{(n)}(x_0),\ f^{(n)}(x_0),\quad \frac{d^n y}{dx^n}(x_0), \frac{d^n f(x_0)}{dx^n},\ f^{(n)}(x)\Big|_{x=x_0}.$$

Proposition 3.1.

If the nth derivative of $f(x)$ is finite at x_0, then there is a neighborhood of x_0 where both $f(x)$ and its first $(n-1)$ derivatives are defined and continuous.

Remark 3.11

1. Proposition 3.1 can be proved by use of the necessary condition for existence of a finite derivative of a function.

2. Higher order derivatives at endpoints of a closed interval are determined via one-sided derivatives of an appropriate order.

Remark 3.12

1. The first derivative y' of the linear function $y = kx + b$ is equal to the constant k, i.e. $y' = k$. All derivatives of order greater than one, the second derivative, the third derivative and so on, are equal to zero: $y'' = y''' = ... = y^{(n)} = 0, n \geq 2$.

2. The first derivative y' of the quadratic function $y = ax^2 + bx + c$ is a linear function $y' = 2ax + b$. The second derivative y'' is a constant: $y'' = 2a$. So all derivatives of order greater than two are equal to zero: $y''' = ... = y^{(n)} = 0, n \geq 3$.

Thus, in general it can be proved that all derivatives of order greater than n of a polynomial of degree n $Q_n(x)$ are identical to zero: $(Q_n(x))^{(n+1)} = (Q_n(x))^{(n+2)} = ... = 0 \ \ \forall x \in R.$

Example 3.18. Derive the formula for the nth derivative $y^{(n)}$, if $y = \ln(1 + x)$.

□ We need to differentiate recursively for finding a pattern. For $x > -1$ we have:

$$y' = (\ln(1 + x))' = \frac{1}{1 + x} = (1 + x)^{-1};$$

$$y'' = (y')' = ((1 + x)^{-1})' = (-1) \cdot (1 + x)^{-2};$$

$$y''' = (y'')' = ((-1) \cdot (1 + x)^{-2})' = (-1) \cdot (-2) \cdot (1 + x)^{-3};$$

....

$$y^{(n)} = (-1) \cdot (-2) ... \cdot (-n + 1) \cdot (1 + x)^{-n} = \frac{(-1)^{n-1} \cdot (n - 1)!}{(1 + x)^n}.$$

The final formula is $(\ln(1 + x))^{(n)} = \dfrac{(-1)^{n-1}(n - 1)!}{(1 + x)^n}$, $n \in N$. ■

Example 3.19. Show that the function $y = C_1x^3 + C_2x + C_3$ is a solution of the differential equation $xy''' - y'' = 0$.

□ To get the answer we can use the following algorithm:

1. Find first three derivatives: $y' = 3C_1x^2 + C_2$, $y'' = 6C_1x$, $y''' = 6C_1$.

2. Put the obtained derivatives into the equation: $x \cdot 6C_1 - 6C_1x = 0 \Leftrightarrow 0 \equiv 0$. We have the identity which holds for all $x \in \mathbb{R}$.

Hence any function of the form: $y = C_1x^3 + C_2x + C_3$ is a solution of the equation $xy''' - y'' = 0$. ■

Physical interpretation of the second derivative

Let $S(t)$ be a distance, traveled by a point during time t. Earlier we defined an acceleration $a(t)$ of an object as the first derivative of a velocity:

$$a(t) = v'(t).$$

Hence, since $v(t) = S'(t)$, the acceleration is equal to the second derivative of $S(t)$ with respect to time t:

$$a(t) = (S'(t))' = S''(t).$$

The acceleration is one of basic characteristics of the motion. If the acceleration is equal to zero over time interval, then such a motion is called a ***uniform motion***. If the acceleration is a positive constant over given time interval, then this motion is called ***uniformly accelerated motion***. If the acceleration is a negative constant over time interval, then the motion is called ***uniformly retarded motion***.

Example 3.20. Find the acceleration of the point, if its motion is represented by $S(t) = 3t^2 + 4t - 5$.

□ Applying $a(t) = (S'(t))' = S''(t)$, we have $S'(t) = 6t + 4 \Rightarrow a(t) = (S'(t))' =$ $= (6t + 4)' = 6$. Thus, the acceleration is a positive constant, so we conclude the motion is uniformly accelerated motion. ■

Higher order differentials

Let $y = f(x)$ be a function which has first n finite derivatives at each point of an interval X. We know

$$d f(x) = f'(x)dx. \tag{3.6}$$

$d f(x)$ is called the ***first differential*** or the differential of order one. The value of the first differential at $x = x_0$ is

$$d f(x_0) = f'(x_0)dx.$$

The differential of the first differential is called the ***second differential*** or the differential of order two of $f(x)$ at x_0:

$$d^2 f(x_0) = d(df(x))\big|_{x=x_0}.$$

If we think of $df(x)$ as a function of the independent variable x and use (3.6), we get

$$d(df(x)\big|_{x=x_0} = (df(x))'\big|_{x=x_0} \cdot dx = (f'(x)dx)'\big|_{x=x_0} \cdot dx = f''(x_0)dx^2.$$

Hence, we have

$$d^2 f(x_0) = f''(x_0)dx^2.$$

If x is any point in X, then $d^2 f(x) = f''(x)dx^2$ or $d^2 y = y'' \cdot dx^2$. Therefore, $y'' = \dfrac{d^2 y}{dx^2}$.

the differential of the second differential is called the ***third differential*** or the differential of order three of $f(x)$ at x_0:

$$d^3 f(x_0) = d(d^2 f(x))\Big|_{x=x_0}.$$

By analogy, if x is an independent variable, then

$$d(d^2 f(x)\Big|_{x=x_0} = (d^2 f(x))'\Big|_{x=x_0} \cdot dx = (f''(x)dx)'\Big|_{x=x_0} \cdot dx = f'''(x_0)dx^3.$$

In this manner, the differential of the $(n-1)$st differential is called the ***nth differential*** or the differential of order n of $f(x)$ at x_0:

$$d^n f(x_0) = d(d^{n-1} f(x))\Big|_{x=x_0}.$$

Summarizing the obtained results, we get the connection between the nth derivative and the nth differential:

$$d^n f(x_0) = f^{(n)}(x_0)dx^n. \qquad (3.7)$$

We can rewrite it as:

$$d^n y = y^{(n)} \cdot dx^n.$$

Hence, $y^{(n)} = \dfrac{d^n y}{dx^n}$. So we have the notation for the nth derivative written in terms of differentials (***Leibniz notation***).

Remark 3.13

In contrast to the notation for the first differential given by (3.6), the notation for higher order differentials $(n \geq 2)$ depends on how we treat x as an independent variable or a function.

Def.: Suppose a function $f(x)$ is such that derivatives $f'(x), f''(x), f'''(x), .., f^{(n)}(x)$ exist at x_0. Then $f(x)$ is called n***-times differentiable at*** x_0.

Def.: A function $f(x)$, which is n-times differentiable at each point of an interval, is called n***-times differentiable on this interval***.

Example 3.21. Find the third derivative y''' and the third differential $d^3 y$, if $y = e^x(x-3)$. Calculate y''' and $d^3 y$ at $x = 3$.

□ Differentiating recursively we get

$$y' = (e^x(x-3))' = e^x(x-3) + e^x \cdot 1 = e^x(x-2);$$
$$y'' = (e^x(x-2))' = e^x(x-2) + e^x = e^x(x-1);$$
$$y''' = (e^x(x-1))' = e^x(x-1) + e^x = e^x x.$$

Then, by (3.7) for $n=3$ we have $d^3y = y'''(dx)^3 = e^x x\,(dx)^3$.

Substituting $x=3$ gives us

$$y'''(3) = 3e^3,$$

$$d^3y(3) = 3e^3(dx)^3.$$

Note, the third derivative at $x=3$ is a constant, but the third differential at this point is a function of dx.■

Higher order derivatives of a function represented parametrically

Let a function y as a function of a variable x be defined by parametric equations:

$$x = \varphi(t), \quad y = \psi(t).$$

If functions $\varphi(t)$ and $\psi(t)$ have higher order derivatives at a point t_0, then there exist corresponding higher order derivatives of the function y with respect to x at this point and these derivatives are defined by:

$$\frac{d^2y}{dx^2} = \frac{(y'_x)'_t}{x'_t} \text{ or } y''_{xx} = \frac{y''_{tt}x'_t - x''_{tt}y'_t}{\left(x'_t\right)^3}, \tag{3.8}$$

where $x'_t = \varphi'(t),\quad x''_{tt} = \varphi''(t),\quad y'_t = \psi'(t),\quad y''_{tt} = \psi''(t)$.

For the third order derivative we have

$$\frac{d^3y}{dx^3} = \frac{(y''_{xx})'_t}{x'_t}.$$

In this manner, we can evaluate derivatives of any order of a function represented parametrically.

Example 3.22. Find y''_{xx}, if the function $y(x)$ is represented by the parametric equations: $\begin{cases} x = 1+\cos t, \\ y = \sin t. \end{cases}$

□ According to the result of example 3.14 $y'_x = -\cot t$. Applying (3.8), we get

$$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{(-\cot t)'_t}{(1+\cos t)'_t} = \frac{1}{\sin^2 t\cdot(-\sin t)} = -\frac{1}{\sin^3 t}.$$ ■

3.11. APPLICATIONS OF DERIVATIVES. L'HÔPITAL'S RULE

Theorem 3.12 (L'Hôpital's Rule, the $\frac{0}{0}$ case).

Let

1) *functions* $f(x)$ *and* $g(x)$ *be defined and differentiable on a punctured neighborhood of* x_0;
2) $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$;
3) $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ *exists.*

Then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Theorem 3.13 (L'Hôpital's Rule, the $\frac{\infty}{\infty}$ case).

Let

1) *functions* $f(x)$ *and* $g(x)$ *be defined and differentiable on a punctured neighborhood of* x_0;
2) $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = \infty$;
3) $\lim\limits_{x\to x_0} \dfrac{f'(x)}{g'(x)}$ *exists.*

Then,

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

Remark 3.14

1. The L'Hôpital's Rule is applicable in the case when
   a) $x \to x_0$, where x_0 is any real number;
   b) $x \to \infty$;
   c) $x \to x_0 \pm 0$, $x \to \pm\infty$.
2. The L'Hôpital's Rule can be also applied to investigate indeterminate forms such as $[\infty - \infty]$, $[1^\infty]$, $[0^\infty]$, $[\infty^0]$. How we should work with the mentioned indeterminate forms to convert them into the $\frac{0}{0}$ case or the $\frac{\infty}{\infty}$ case is given below:

- $[0 \cdot \infty]$. We have $\lim\limits_{x\to a} u \cdot v = [0 \cdot \infty]$. Then, the product $u \cdot v$ is represented by the quotient:

$$u\cdot v=[0\cdot\infty]=\begin{bmatrix}\dfrac{u}{(1/v)}=\left[\dfrac{0}{0}\right],\\ \dfrac{v}{(1/u)}=\left[\dfrac{\infty}{\infty}\right]\end{bmatrix}.$$

➢ $[\infty-\infty]$. We have $\lim\limits_{x\to x_0} u-v=[\infty-\infty]$. If $u\neq 0$ we can factor out u:

$$u-v=u\cdot\left(1-\frac{v}{u}\right),$$

Or if $v\neq 0$ we can factor out v:

$$u-v=v\cdot\left(\frac{u}{v}-1\right).$$

Then,

- if $\lim\limits_{x\to x_0}\dfrac{v}{u}$ exists and $\lim\limits_{x\to x_0}\dfrac{v}{u}=A\neq 1$, then $\lim\limits_{x\to x_0}(u-v)=\lim\limits_{x\to x_0}u\,(1-A)=\infty$;
- if $\lim\limits_{x\to x_0}\dfrac{v}{u}$ exists and $\lim\limits_{x\to x_0}\dfrac{v}{u}=1$, then $\lim\limits_{x\to x_0}(u-v)=\lim\limits_{x\to x_0}\dfrac{1-\dfrac{v}{u}}{\dfrac{1}{u}}=\left[\dfrac{0}{0}\right]$.

Further, obviously we apply the L'Hôpital's Rule in straight forward manner.

➢ $[1^\infty]$, $[0^\infty]$, $[\infty^0]$. We have $\lim\limits_{x\to x_0} u^v=\begin{bmatrix}[1^\infty],\\ [0^\infty],\\ [\infty^0].\end{bmatrix}$ Using the fact that $f(x)=e^{\ln f(x)}$, we get

$$\lim_{x\to x_0} u^v=\lim_{x\to x_0} e^{\ln u^v}=\left|\ln u^v=v\ln u\right|=\lim_{x\to x_0} e^{v\ln u}=e^{[0\cdot\infty]}.$$

To investigate the indeterminate form $[0\cdot\infty]$ the L'Hôpital's Rule is used as described above.

Example 3.23. Find $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}+\sqrt[3]{x+8}-3}{x}$.

□ $\lim\limits_{x\to 0}\dfrac{\sqrt{x+1}+\sqrt[3]{x+8}-3}{x}=\left[\dfrac{0}{0}\right]=\lim\limits_{x\to 0}\dfrac{(\sqrt{x+1}+\sqrt[3]{x+8}-3)'}{x'}=$

$$=\lim_{x\to 0}\left(\frac{1}{2\sqrt{x+1}}+\frac{1}{3\sqrt[3]{(x+8)^2}}\right)=\frac{7}{12}.\ ■$$

Example 3.24. Find $\lim\limits_{x\to 0+0}\dfrac{\ln x}{\cot x}$.

□ $\lim\limits_{x\to 0+0}\frac{\ln x}{\cot x}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to 0+0}\frac{(\ln x)'}{(\cot x)'}=\lim\limits_{x\to 0+0}\frac{1\cdot\sin^2 x}{x\cdot(-1)}=-\lim\limits_{x\to 0+0}\frac{\sin^2 x}{x}=\left[\frac{0}{0}\right]=$

$$=-\lim_{x\to 0+0}\frac{(\sin^2 x)'}{x'}=-\lim_{x\to 0+0}\frac{2\sin x\cdot\cos x}{1}=0. \blacksquare$$

The L'Hôpital's Rule can be applied as many times as functions under the limit sign satisfy all hypothesis of the given above theorems.

Example 3.25. Find $\lim\limits_{x\to+\infty}(\ln x)^{\frac{1}{x}}$.

□ $\lim\limits_{x\to+\infty}(\ln x)^{\frac{1}{x}}=\left[\infty^0\right]=\lim\limits_{x\to+\infty}\left(e^{\ln\ln x}\right)^{\frac{1}{x}}=\lim\limits_{x\to+\infty}e^{\frac{\ln\ln x}{x}}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to+\infty}e^{\frac{(\ln\ln x)'}{x'}}=$

$$=\lim_{x\to+\infty}e^{\frac{1}{\ln x\cdot x}}=e^0=1. \blacksquare$$

Example 3.26. Find the following limits:

a) $\lim\limits_{x\to 1}\frac{\ln(2-\sqrt[5]{x})}{x^2-x}$; b) $\lim\limits_{x\to 0+0}\frac{\ln x}{\lg x}$; c) $\lim\limits_{x\to+\infty}\frac{2^x}{x^2}$.

□ a) $\lim\limits_{x\to 1}\frac{\ln(2-\sqrt[5]{x})}{x^2-x}=\left[\frac{0}{0}\right]=\lim\limits_{x\to 1}\frac{(\ln(2-\sqrt[5]{x}))'}{(x^2-x)'}=\lim\limits_{x\to 1}\frac{-\frac{1}{5}x^{\frac{4}{5}}}{(2-\sqrt[5]{x})(2x-1)}=$

$=-\frac{1}{5\cdot 1\cdot 1}=\frac{1}{5}$;

b) $\lim\limits_{x\to 0+0}\frac{\ln x}{\lg x}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to 0+0}\frac{(\ln x)'}{(\lg x)'}=\lim\limits_{x\to 0+0}\frac{1/x}{1/x\cdot\ln 10}=\ln 10.$

c) $\lim\limits_{x\to+\infty}\frac{2^x}{x^2}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to+\infty}\frac{(2^x)'}{(x^2)'}=\lim\limits_{x\to+\infty}\frac{2^x\ln 2}{2x}=\left[\frac{\infty}{\infty}\right]=\lim\limits_{x\to+\infty}\frac{(2^x\ln 2)'}{(2x)'}=$

$=\lim\limits_{x\to+\infty}\frac{2^x\ln^2 2}{2}=+\infty$.■

Example 3.27. Find $\lim\limits_{x\to\infty}\frac{x+\sin x}{x}$.

□ As $x\to\infty$, the numerator and the denominator of the fraction $\frac{x+\sin x}{x}$

increase unboundedly. So we deal with the indeterminate form $\left[\frac{\infty}{\infty}\right]$. Unfortunately, we can not use the L'Hôpital's Rule: there is no limit $\lim\limits_{x\to+\infty}\frac{(x+\sin x)'}{x'}=\lim\limits_{x\to+\infty}\frac{1+\cos x}{1}$, because $\cos x$ as well as $1+\cos x$ have no limits at infinity. However, the limit $\lim\limits_{x\to\infty}\frac{x+\sin x}{x}$ exists and can be found in another way:

$\lim\limits_{x\to\infty}\frac{x+\sin x}{x}=\lim\limits_{x\to\infty}\left(1+\sin x\cdot\frac{1}{x}\right)=1$. We should divide each term of the numerator by the denominator. The limit of the second summand is equal to zero as the limit of a product of the bounded function $\sin x$ and the infinitesimal function $\frac{1}{x}$, $x\to\infty$ (see properties of infinitesimals). ■

Exercises

1. Find derivatives of appropriate orders:

a) $y^{(5)}$ if $y=(2x^2-7)\ln(x-1)$;

b) y''' if $y=\frac{\log_2 x}{x^3}$;

c) $y^{(4)}$ if $y=e^{x/2}\cdot\sin 2x$.

2. Find y''_{xx} if $y(x)$ is given by

a) $\begin{cases}x=\cos 2t,\\ y=2\sec^2 t;\end{cases}$

b) $\begin{cases}x=1/t,\\ y=1/(1+t^2);\end{cases}$

c) $\begin{cases}x=e^t\cos t,\\ y=e^t\sin t.\end{cases}$

In the given below exercises a student's number and the last numeral in a group number should be taken for m and n respectively.

3. Find the limits using the L'Hôpital's Rule:

a) $\lim\limits_{x\to+\infty}\frac{\ln nx}{mx^2}$;

b) $\lim\limits_{x\to0}\frac{1-\cos nx}{x\cdot\sin x}$;

c) $\lim\limits_{x\to0}\frac{n\sqrt[3]{x+1}-\sqrt{x+n^2}}{x^2-x}$;

d) $\lim\limits_{x\to a}\frac{x^3-nx^2+x-n}{x^3-nx^2-x+n}$;

e) $\lim\limits_{x\to0+0}xe^{\frac{m}{x}}$;

f) $\lim\limits_{x\to a}\left(\frac{x}{n}\right)^{\frac{1}{x-n}}$;

g) $\lim\limits_{x\to a}(x-n)^{x^2-n^2}$

3.12. APPLICATIONS OF DERIVATIVES. SKETCHING GRAPHS

Let $f(x)$ be defined on an interval X.

Def.: $f(x)$ is said to be ***increasing*** (***decreasing***) on X, if $f(x_1) \geq f(x_2)$ ($f(x_1) \leq f(x_2)$) whenever $x_1 > x_2$, $x_1, x_2 \in X$.

Def.: $f(x)$ is said to be ***strictly increasing*** (***decreasing***) on X, if $f(x_1) > f(x_2)$ ($f(x_1) < f(x_2)$) whenever $x_1 > x_2$, $x_1, x_2 \in X$.

Def.: $f(x)$ that is increasing or decreasing on X is called ***monotonic*** on X.

Def.: $f(x_0)$ is called a ***local maximum*** (***local minimum***) if there exists a neighborhood $U(x_0)$ of x_0 such that $f(x) \geq f(x_0)$; ($f(x) \leq f(x_0)$) for all x in $U(x_0)$.

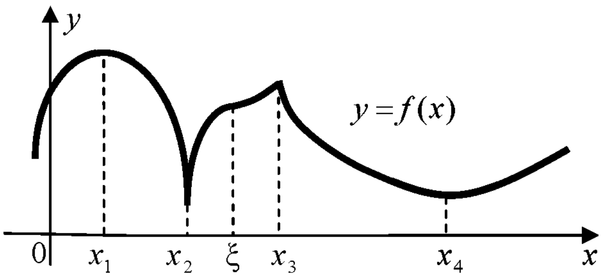
Def.: A ***local extremum*** is called either a local minimum or a local maximum.

Further we use just maximum (minimum) instead of a local maximum (minimum).

Remark 3.15

If a function $f(x)$ has a maximum (minimum) at x_0, then the point $(x_0, f(x_0))$ is called a ***maximum*** (minimum) ***point***.

In pic. 3.9 there is shown the graph of $f(x)$.



Pic. 3.9

Function $f(x)$ is increasing on the intervals $(-\infty; x_1]$, $[x_2; x_3]$, $[x_4; \infty)$ and decreasing on the intervals $[x_1; x_2]$, $[x_3; x_4]$. $(x_1, f(x_1))$, $(x_3, f(x_3))$ are maximum points, $(x_2, f(x_2))$, $(x_4, f(x_4))$ are minimum points. Obviously, if we have a graph we are able to indicate, for example, intervals of increasing $f(x)$, enough simply.

Proposition 3.2

A function $f(x)$ is constant on X if and only if $f'(x) = 0$ for every x in X.

$$f(x) = \text{const } \forall x \in X \Leftrightarrow f'(x) = 0 \; \forall x \in X.$$

Remark 3.16

$f(x) = \text{const}$ is a function which is both increasing and decreasing on $\mathbb{R}$.

Theorem 3.14 (Increasing/Decreasing Test, the First Derivative Test for monotonic functions).

If $f'(x)>0$ on X, then $f(x)$ is strictly increasing on X.
If $f'(x)<0$ on X, then $f(x)$ is strictly decreasing on X.

Using mathematical symbols, the above statements can be written in the form:

$$\forall x\in X \; f'(x)>0 \Rightarrow \forall x_1,x_2\in X,\; x_2>x_1,\; f(x_2)>f(x_1).$$

$$\forall x\in X \; f'(x)<0 \Rightarrow \forall x_1,x_2\in X,\; x_2>x_1,\; f(x_2)<f(x_1).$$

Theorem 3.15

If a function $f(x)$ has an extremum at a point x_0, then $f'(x_0)=0$ or $f'(x_0)$ does not exist.

Remark 3.17

" $f'(x_0)$ does not exist " means that a finite derivative does not exist.

For example, $f(x)$ has a minimum at the point x_2 and $f'(x_2)=\infty$ (see pic. 3.9). $f(x)$ has no finite derivative at this point. $f(x)$ has a maximum at the point x_3 and one-sided derivatives exist, but are not equal. So we get again that $f(x)$ has no finite derivative at the point.

Def.: A point x_0 in X is called a ***critical point*** of $f(x)$, if $f'(x_0)=0$ or $f'(x_0)$ does not exist. A point where $f'(x_0)=0$ is called a ***stationary point***.

Remark 3.18

A function does not always take on an extreme value at a critical point. For example, in pic. 3.9 there is the point ξ such that $f'(\xi)=0$. But ξ is not a minimum or maximum point.

Theorem 3.16 (The First Derivative Test for Extrema).

Suppose, $x_0\in X$ is a critical point of $f(x)$ and $f(x)$ is differentiable on a punctured neighborhood $\overset{o}{U}(x_0)$ of x_0.

If $f'(x)>0$ for $x<x_0, x\in\overset{o}{U}(x_0)$ and $f'(x)<0$ for $x>x_0, x\in\overset{o}{U}(x_0)$ then $(x_0,f(x_0))$ is a maximum point of $f(x)$.

If $f'(x)<0$ for $x<x_0, x\in\overset{o}{U}(x_0)$ and $f'(x)>0$ for $x>x_0, x\in\overset{o}{U}(x_0)$ then $(x_0,f(x_0))$ is a minimum point of $f(x)$.

If $f'(x)>0$ or $f'(x)<0$ for all $x\in\overset{o}{U}(x_0)$, then $(x_0,f(x_0))$ is not extreme point of $f(x)$.

Theorem 3.17

Suppose, $x_0 \in X$ *is a stationary point of* $f(x)$ *and derivatives* $f'(x), f''(x), f'''(x),.., f^{(n+1)}(x)$ *exist in a neighborhood* $U(x_0)$ *of* x_0*. Suppose,* $f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0$, and $f^{(n)}(x_0) \neq 0$.

If n *is an even number and* $f^{(n)}(x_0) > 0$*, then* $f(x)$ *has a minimum at* x_0*.*

If n *is an even number and* $f^{(n)}(x_0) < 0$ *, then* $f(x)$ *has a maximum at* x_0*.*

If n *is an odd number, then* $f(x)$ *has no extremum at* x_0*.*

Corollary (the Second Derivative Test for Extrema)

If $f''(x_0) > 0$ *at a stationary point* x_0*, then* $f(x)$ *has a minimum at* x_0*.*

If $f''(x_0) < 0$*, at a stationary point* x_0*, then* $f(x)$ *has a maximum at* x_0*.*

Remark 3.19

We use the Second Derivative Test for Extrema instead of the First Derivative Test, when it is more difficult to investigate sign changes in f' at x_0, than to find higher order derivatives.

Algorithm for investigating a function for extrema and indicating intervals where the function is increasing/decreasing

1. Find the domain $D(f)$ of $f(x)$.
2. Find $f'(x)$.
3. Find critical points $x_0 \in D(f)$, i.e. points where $f' = 0$ or f' does not exist.
4. Use the First Derivative Test to determine extreme points and intervals on which a function is increasing and decreasing.
5. Find extreme values of a function.

Example 3.28. Find extema of $f(x)$ and intervals where $f(x)$ is increasing/decreasing, if $f(x) = e^{-x^2}$.

□ Applying the algorithm given above,

1. $f(x)$ is defined, continuous and differentiable on $\mathbb{R}$: $D(f) = \mathbb{R}$.

2. $f'(x) = -2xe^{-x^2}$.

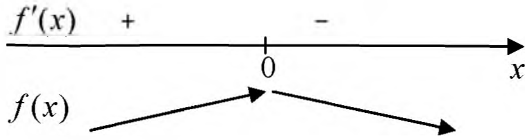
3. $f'(x) = 0$ at $x = 0$. Hence, $x_0 = 0$ is a stationary point of $f(x)$. Since $f'(x)$ is continuous on $\mathbb{R}$ there are no points where $f'(x)$ is undefined (doesn't exist).

4. $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. Thus, according to the Increasing/Decreasing Test $f(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$. Moreover, moving across the point $x = 0$ from left to right $f'(x)$ changes from positive to negative, so following the First Derivative Test for Extrema $f(x)$ has a maximum at $x = 0$.

5. $f_{\max} = f(0) = 1$. ■

Remark 3.20

We may use the scheme as shown in pic. 3.10 to examine behavior of $f(x)$ and find extreme points. The stationary point $x=0$ divides the x-axis into two intervals: $(-\infty,0)$, $[0,+\infty)$. To determine sign of $f'(x)$ we may use a test point. The test point is any point in the given intervals. For example, the test point for



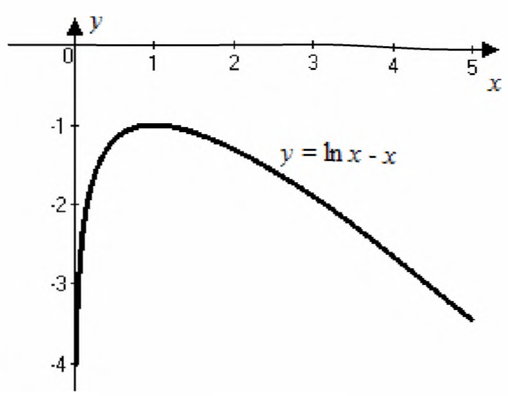
Pic. 3.10

$(-\infty,0)$ is -1. Then, we calculate the value $f'(-1)=-2\cdot(-1)\cdot e^{-(-1)^2}=\dfrac{2}{e}>0$. By analogy, the test point for $[0,+\infty)$ is 1. Then, $f'(1)=-2\cdot 1\cdot e^{-1^2}=-\dfrac{2}{e}<0$. We mark it above corresponding intervals on the scheme. Behavior of the function on these intervals is shown with arrows under the x-axis.

Example 3.29. Find extrema points of $f(x)$ and intervals on which $f(x)$ is increasing and decreasing, if $f(x)=\ln x-x$.

□ According to the algorithm

1. $f(x)$ is defined, continuous and infinitely differentiable for $x>0$: $D(f)=(0;+\infty)$.
2. $f'(x)=(\ln x-x)'=\dfrac{1}{x}-1=\dfrac{1-x}{x}$.
3. $f'(x)=0 \Leftrightarrow x-1=0 \Leftrightarrow x=1$. Hence $x=1$ is a stationary point. There are no other critical points because $x=0$ is out of the domain.
4. Since $x>0$, the sign of $f'(x)$ is determined by the sign of the difference $(1-x)$. Therefore, $f'(x)>0$ for $x\in(0;1)$ and $f'(x)<0$ for $x>1$. So we can draw the following conclusion: the function increases for $x\in(0;1)$ and decreases for $x>1$, thus $f(x)$ has a global maximum at $x=1$(pic. 3.11).
5. $f_{\max}=f(1)=-1$. ■



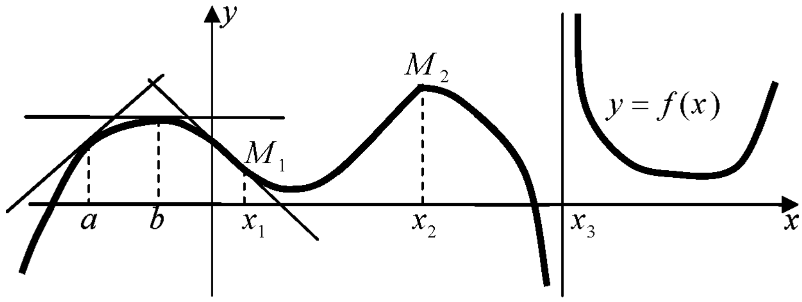
Pic. 3.11

Concavity and points of inflection

Let a function $f(x)$ be differentiable on an interval X.

Def.: The graph of $f(x)$ is called ***concave upward*** (***concave downward***) if all points of the graph are above (below) any tangent line to this graph on X.

Pic. 3.12 illustrates the graph of $f(x)$ which is *concave upward* on the intervals: $[x_1;\ x_2]$, $[x_3; \infty)$ and *concave downward* on the intervals: $(-\infty;\ x_1]$, $[x_2; x_3]$.



Pic. 3.12

Indeed, let's consider the interval $(-\infty;\ x_1]$, containing two points a and b. The graph of $f(x)$ lies below tangent lines through the points $(a, f(a))$, $(b, f(b))$. So by the definition the graph is concave downward on $(-\infty;\ x_1]$. By analogy, we can illustrate given above conclusion about concavity of the graph on each intervals.

Def.: A point $M_0(x_0,\ f(x_0))$ on the graph of $f(x)$ where concavity changes is called a ***point of inflection***.

In pic. 3.12 x_1, x_2 are *points of inflection*. Let's show it for the point x_1. The graph is concave downward for points which are prior to the point x_1 and it is concave upward for points which are after x_1. Thus, as x increases through x_1 concavity changes from downward to upward. x_3 is not a point of inflection. Although concavity changes from downward to upward, x_3 is not in the domain of $f(x)$.

Remark 3.21

In pic. 3.12 there are three tangent lines to the graph on $(-\infty;\ x_1]$. The graph is concave downward on it. Note, the slopes of these tangent lines decreases as x increases. The slopes are determined by values of the derivative $f'(x)$. So, *if the derivative $f'(x)$ decreases as x increases, then the graph is concave downward.* For concavity upward we have: *if $f'(x)$ increases as x increases, then the graph is concave upward.*

Theorem 3.18 (The Second Derivative Test for Concavity).

If $f''(x) > 0$ ($f''(x) < 0$) *on an interval* X, *then the graph of* $f(x)$ *is concave upward (downward) on* X.

Theorem 3.19

If x_0 *is a point of inflection, then* $f''(x_0) = 0$ *or* $f''(x_0)$ *does not exist.*

Remark 3.22

1. “$f''(x_0)$ does not exist ” means that finite derivative does not exist.

2. Points satisfying above theorem may not be points of inflection. This theorem gives only the necessary condition, but not the sufficient condition.

Theorem 3.20

Suppose, $x_0 \in D(f)$ *and* $f''(x_0) = 0$ *or* $f''(x_0)$ *does not exist.*

Then, if $f''(x)$ *changes sign as x increases through* x_0, *then* x_0 *is a point of inflection.*

Remark 3.23

According to theorem, if x_0 is a critical point, but not an extreme point, then x_0 is a point of inflection.

Algorithm for identifying points of inflection and concavity of the graph

1. Find the domain $D(f)$ of $f(x)$.
2. Find $f''(x)$.
3. Find points $x_0 \in D(f)$ such that $f'' = 0$ or f'' does not exist.
4. Examine wherever $f''(x) > 0$ or $f''(x) < 0$ on both sides of each points x_0. Determine intervals where the graph of $f(x)$ is concave upward and downward.
5. Find points of inflection and values of $f(x)$ at these points.

Example 3.30. Find points of inflection and identify intervals on which the graph of $f(x)$ is concave upward or concave downward, if $f(x) = e^{-x^2}$.

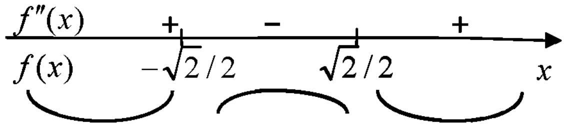
□ Applying the algorithm given above

1. $f(x)$ is defined, continuous and differentiable on $\mathbb{R}$: $D(f) = \mathbb{R}$.

2. $f''(x) = (-2xe^{-x^2})' = -2e^{-x^2} + \cdot(-2x)e^{-x^2}(-2x) = 2e^{-x^2}(2x^2 - 1)$.

3. $f''(x) = 0 \Leftrightarrow 2e^{-x^2}(2x^2 - 1) = 0 \Leftrightarrow x = \pm\frac{\sqrt{2}}{2}$.

4. $f''(x) > 0$ on $\left(-\infty; -\frac{\sqrt{2}}{2}\right] \cup \left[\frac{\sqrt{2}}{2}; \infty\right)$, so the graph is concave upward. $f''(x) < 0$ on $\left[-\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right]$, consequently the graph is concave downward.



Pic. 3.13

$f''(x)$ changes sign twice as x increases through $x=-\dfrac{\sqrt{2}}{2}$ and $x=\dfrac{\sqrt{2}}{2}$. Thus, these points are points of inflection.

5. $f\left(\dfrac{\sqrt{2}}{2}\right)=f\left(-\dfrac{\sqrt{2}}{2}\right)=e^{-\frac{1}{2}}=\dfrac{1}{\sqrt{e}}$. ■

Remark 3.24

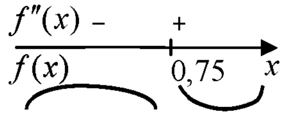
We may illustrate the solution with the scheme shown in pic. 3.13. Points $-\dfrac{\sqrt{2}}{2}$ and $\dfrac{\sqrt{2}}{2}$ divide the x-axis into three intervals: $\left(-\infty;-\dfrac{\sqrt{2}}{2}\right]$, $\left[-\dfrac{\sqrt{2}}{2};\dfrac{\sqrt{2}}{2}\right]$, $\left[\dfrac{\sqrt{2}}{2};\infty\right)$. Above the x-axis we mark signs of the second derivative on each of these intervals, below it we show concavity of the graph with symbols ($\cup$), ($\cap$).

Example 3.31. Find intervals, where the graph is concave downward and upward, and points of inflection if $f(x)=4x^3-9x^2-12x+8$.

□ 1. $D(f)=\mathbb{R}$

2. $f''(x)=(12x^2-18x-12)'=24x-18$.

3. $f''(x)=0 \Leftrightarrow 24x-18=0 \Leftrightarrow x=0{,}75$.



Pic. 3.14

4. The sign of the second derivative:

$$\left[\begin{array}{l} f''(x)>0 \Leftrightarrow x>0{,}75 \\ f''(x)<0 \Leftrightarrow x<0{,}75 \end{array}\right.$$ (see pic.3.14).

The function f is concave upward for $x\in(0{,}75;\infty)$ and concave downward for $x\in(-\infty;\ 0{,}75\ )$. The point $x=0{,}75$ is a point of inflection.

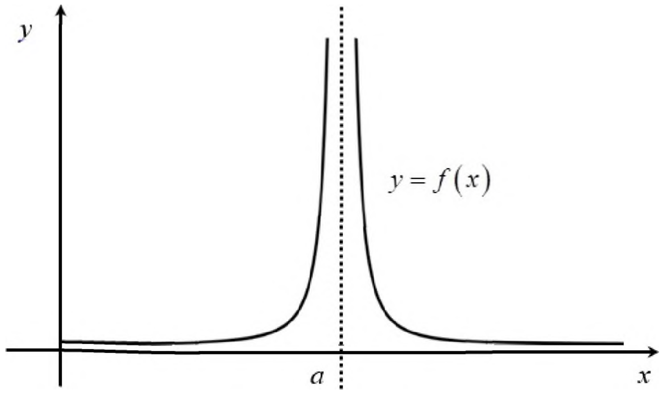
5. $f(0{,}75)=-4{,}375$. The coordinates of the point of inflection are $(0{,}75;\ -4{,}375)$. ■

Asymptotes

Def.: An ***asymptote*** for the graph of $f(x)$ is called a straight line such that a distance between points $(x, f(x))$ on the graph and points on the line goes to zero as x increases without bound.

Def.: A vertical line $x=a$ is called a ***vertical asymptote*** for the graph of $f(x)$, if at least one of the following limits $\lim\limits_{x\to a-0} f(x)$ and/or $\lim\limits_{x\to a+0} f(x)$ are equal to infinity, i.e. $\left[\begin{array}{l} \lim\limits_{x\to a-0} f(x)=\infty, \\ \lim\limits_{x\to a+0} f(x)=\infty. \end{array}\right.$

In pic. 3.15 there is an example of the graph of a function, which has a vertical asymptote $x = a$.

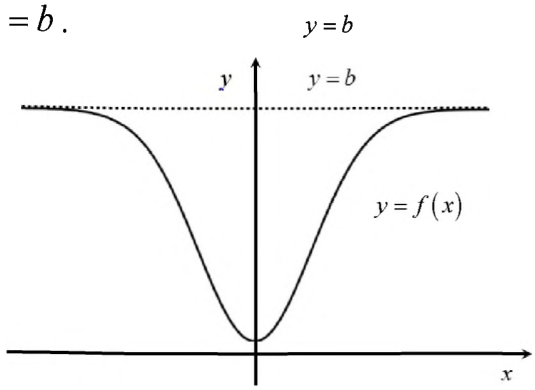


Pic. 3.15

Def.: ***Def.***: Let a function $f(x)$ be defined for all $x > K$, $K > 0$. A horizontal line $y = b$ is called a ***horizontal asymptote*** (as $x \to +\infty$), if there exists the finite limit $\lim\limits_{x \to +\infty} f(x) = b$.

Now, let a function $f(x)$ be defined for all $x < K$, $K < 0$. If there exists the finite limit $\lim\limits_{x \to -\infty} f(x) = b$, then, as before, a horizontal line $y = b$ is called a ***horizontal asymptote*** (as $x \to -\infty$).

In pic. 3.16 there is presented the example of the graph of a function which has a horizontal asymptote $y = b$.



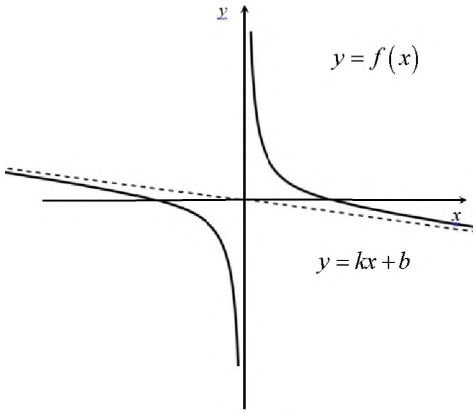
Pic. 3.16

Def.: Let a function $f(x)$ be defined for all $x > K$, $K > 0$. Suppose, $f(x)$ can be represented by $f(x) = kx + b + o(x)$ as $x \to +\infty$. $o(x)$ is infinitesimal of higher order than x. Then, a line $y = kx + b$ is called an ***oblique*** or a ***slant asymptote*** as $x \to +\infty$. By analogy, also we can define an oblique asymptote as $x \to -\infty$.

Theorem 3.21

Let a function $f(x)$ be defined for all $x > K$, $K > 0$. A line $y = kx + b$ is an oblique asymptote if and only if there exist finite limits $k = \lim\limits_{x \to +\infty} \frac{f(x)}{x}$ and $b = \lim\limits_{x \to +\infty} (f(x) - kx)$.

In pic. 3.17 there is presented the example of the graph of a function which has an oblique asymptote $y = kx + b$.



Pic. 3.17

Example 3.32. Find asymptotes of the graph of $f(x) = e^{-x^2}$.

□ $f(x)$ is one of basic elementary functions. It is defined and continuous on $\mathbb{R}$. So, $f(x)$ has no points of discontinuity and, thus, it has no vertical asymptotes.

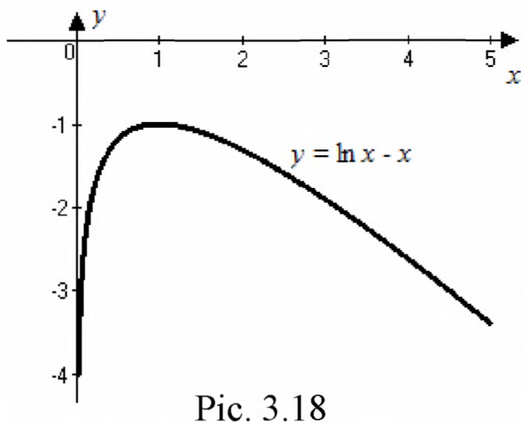
Let's investigate behavior of $f(x)$ as $x \to \infty$: $\lim\limits_{x\to\infty} e^{-x^2} = \lim\limits_{x\to\infty} \frac{1}{e^{x^2}} = 0$. Hence, the line $y = 0$ is a horizontal asymptote.■

Example 3.33. Find asymptotes of the graph of the function $y = \ln x - x$.

□ The function $y = \ln x - x$ is defined, continuous and differentiable on $(0; +\infty)$. The graph has a vertical asymptote $x = 0$, which passes through the boundary point $x = 0$ of the domain, as $\lim\limits_{x\to 0+0} (\ln x - x) = -\infty$ (see pic. 3.18).

The graph has no horizontal asymptote as $\lim\limits_{x\to+\infty} (\ln x - x) = -\infty$.

Let's figure out whether there is an oblique asymptote:



Pic. 3.18

$$k = \lim_{x\to+\infty} \frac{\ln x - x}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to+\infty} \frac{(\ln x - x)'}{x'} =$$

$$\lim_{x\to+\infty} \frac{\frac{1}{x} - 1}{1} = -1.$$

$$b = \lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} (\ln x - x + x) =$$

$$= \lim_{x\to+\infty} \ln x = +\infty.$$

The limit is infinite, so the graph has no oblique asymptotes.■

Example 3.34. Find asymptotes of the graph of the function $y = \dfrac{2x^2 - 3}{x - 4}$.

□ The function $y = \dfrac{2x^2 - 3}{x - 4}$ is defined and continuous on $\mathbb{R} \setminus \{4\}$. The point $x = 4$ is a point of discontinuity and $\lim\limits_{x \to 4} \dfrac{2x^2 - 3}{x - 4} = \left[\dfrac{1}{0}\right] = \infty$. Hence the line $x = 4$ is a vertical asymptote (see pic. 3.19).

The graph has no horizontal asymptote:

$$\lim_{x \to +\infty} \frac{2x^2 - 3}{x - 4} = \left[\frac{\infty}{\infty}\right] = \lim_{x \to +\infty} \frac{(2x^2 - 3)'}{(x - 4)'} =$$

$$= \lim_{x \to +\infty} \frac{4x}{1} = \infty.$$

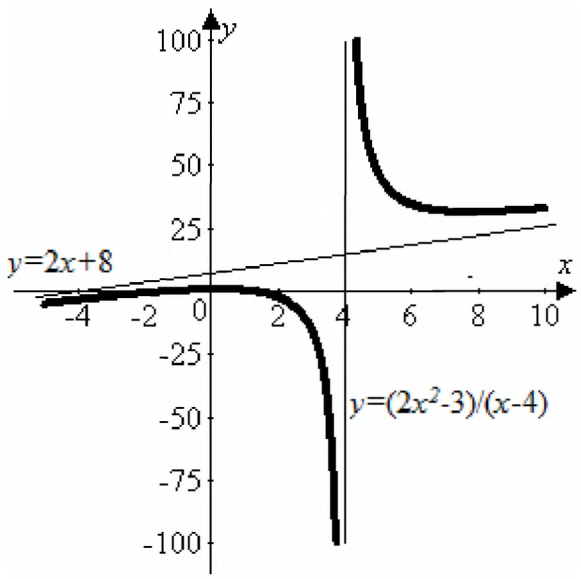
But there is an oblique asymptote $y = 2x + 8$.

Indeed,

$$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{2x^2 - 3}{(x - 4)x} = 2,$$

$$b = \lim_{x \to \infty} (f(x) - kx) =$$

$$= \lim_{x \to \infty} \left(\frac{2x^2 - 3}{x - 4} - 2x \right) = \lim_{x \to \infty} \left(\frac{8x - 3}{x - 4} \right) = 8. \blacksquare$$



Pic. 3.19

Guideline for sketching the graph of a function

1. Identify the domain $D(f)$ of $f(x)$.
2. Find the *x*- and *y*-intercepts and identify intervals, where $f(x)$ doesn't change sign.
3. Identify whether $f(x)$ is an even/odd function or not.
4. Identify whether $f(x)$ is a periodic function or not.
5. Determine intervals on which $f(x)$ is continuous. Find points of discontinuity and vertical asymptotes.
6. Analyze behavior of $f(x)$ at infinity. Find horizontal asymptotes and oblique asymptotes.
7. Determine intervals on which $f(x)$ is increasing and decreasing. Find extreme points.
8. Determine intervals of concavity of $f(x)$. Find points of inflection.
9. Fill in a table with values of $f(x)$, $f'(x)$ and $f''(x)$.
10. Sketch the graph of $f(x)$.

Example 3.35. Investigate the function $y = xe^{-x}$ and sketch the graph.

□ 1. The domain is $D(f) = \mathbb{R}$.

2. There is one point of intersection with the x-axis: $x = 0 \Rightarrow y = 0$, $y = 0 \Rightarrow x = 0$, because $e^{-x} \neq 0 \Rightarrow$ it is the point (0;0). It is easy to analyze the sign of $y(x)$, taking into account that $e^{-x} > 0$ for all x.

3. The domain of function $D(f)$ is symmetric, but the function is neither even nor odd: $f(-x) = -xe^{-(-x)} = -xe^{x} \Rightarrow f(-x) \neq f(x), f(-x) \neq -f(x)$.

4. The function is also not a periodic function: $f(x+T) = (x+T)e^{-(x+T)} \neq f(x)$ for $T \neq 0$.

5. $y = xe^{-x}$ is defined and continuous on $\mathbb{R} \Rightarrow$ there are no vertical asymptotes.

6. Let's examine how the function behaves as $x \to -\infty$ and as $x \to +\infty$.

$\lim\limits_{x\to-\infty} xe^{-x} = [-\infty \cdot \infty] = -\infty \Rightarrow$ there are no horizontal asymptote as $x \to -\infty$.

Let's try to find an oblique asymptote $y = kx + b$ as $x \to -\infty$.

$$k = \lim_{x\to-\infty} \frac{xe^{-x}}{x} = \lim_{x\to-\infty} e^{-x} = +\infty \Rightarrow \text{ as } x \to -\infty.$$

Note, we get the same behavior for k if $x \to +\infty$. Thus, there are no oblique asymptotes

Now let $x \to +\infty$. Applying the L'Hôpital's Rule $\lim\limits_{x\to+\infty} xe^{-x} = [\infty \cdot 0] = \lim\limits_{x\to+\infty} \frac{x}{e^{x}} = \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to+\infty} \frac{1}{e^{x}} = 0$ Therefore, the line $y = 0$ is a horizontal asymptote as $x \to +\infty$.

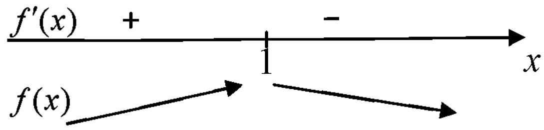
7. Find the first derivative:

$y'(x) = (xe^{-x})' = 1 \cdot e^{-x} + xe^{-x}(-1) = e^{-x}(1-x);$

Further find critical points:

$$\left[\begin{array}{l} y'(x) = 0 \\ y'(x) = \infty \end{array}\right. \quad \left[\begin{array}{l} e^{-x}(1-x) = 0 \\ e^{-x}(1-x) = \infty \end{array}\right. \Rightarrow \left[\begin{array}{l} x = 1, \\ x \in \varnothing. \end{array}\right.$$

Appling the Increasing/Decreasing Test and the First Derivative Test for Extrema (see pic. 3.20), we have

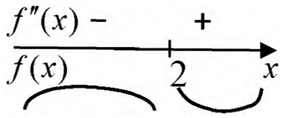


Pic. 3.20

$f(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, +\infty)$. Since $y(1) = 1 \cdot e^{-1} = \frac{1}{e} \approx 0{,}4$, the point $\left(1; \frac{1}{e}\right)$ is a maximum point.

8. Using the second derivative we get:

$$y''(x)=[e^{-x}(1-x)]'=-e^{-x}(1-x)+e^{-x}(-1)=$$
$$=-e^{-x}(1-x+1)=e^{-x}(x-2);$$



Pic. 3.21

$$\left[\begin{array}{l} y''=0 \\ y''=\infty \end{array}\right. \Rightarrow \left[\begin{array}{l} e^{-x}(x-2)=0 \Rightarrow x=2 \\ e^{-x}(x-2)=\infty \Rightarrow x\in\varnothing. \end{array}\right.$$

Then according to the Second Derivative Test for Concavity $f(x)$ is concave upward for $x\in(2;\infty)$ and concave downward for $x\in(-\infty;\ 2)$. Since $y(2)=2e^{-2}=\dfrac{2}{e^2}\approx 0{,}3$, the point $\left(2;\dfrac{2}{e^2}\right)$ is a point of inflection (pic. 3.21).

9. Now we fill in the following table:

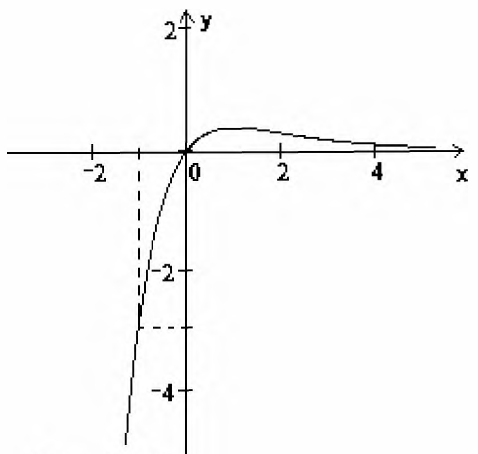
Table 3.1

x	$(-\infty;1)$	1	$(1;2)$	2	$(2;+\infty)$
y'	+	0	–	–	–
y''	–	–	–	0	+
y	↗ ∩	max, $\dfrac{1}{e}$	↘ ∩	infl. point $\dfrac{2}{e^2}$	↘ ∪

The graph of the function $y=xe^{-x}$ is depicted below (pic. 3.22). Obviously, the range of values is $E(f)=\left(-\infty;\dfrac{1}{e}\right]$.

Note that we used one more point while drawing the graph:

$$y(-1)=-e\approx-2{,}7.\blacksquare$$



Pic. 3.22

Exercises

Investigate the functions and sketch the graph.

a) $y=\dfrac{2x}{1+x^2}$; b) $y=\dfrac{2x}{2+x^3}$;

c) $y=xe^{-\frac{x^2}{2}}$; d) $y=\dfrac{\ln x}{x}$;

e) $y=x^2(x-2)^2$; f) $y=\dfrac{x}{2}-\arctan x$.

CHAPTER 4. INTEGRAL CALCULUS

4.1. ANTIDERIVATIVES AND INDEFINITE INTEGRALS

In the previous chapter we have studied how to find a derivative of a function and how to apply it for solving different problems. Now we want to recover a function from its known derivative. In other words starting with f we wish to find a function F whose derivative is f. Such a function is called an anti-derivative.

Suppose, $f(x)$ is a continuous function on X and $F(x)$ is a differentiable function on X.

Def.: The function $F(x)$ is called an ***antiderivative*** of $f(x)$ on X if $F'(x)=f(x)$ for all $x \in X$.

For example, $F(x)=x^2$ is an antiderivative of $f(x)=2x$ as $(x^2)'=2x$ for any real x. At the same time $(x^2+1)'=2x$ and $(x^2-1000)'=2x$. So we can say the functions x^2+1 and x^2-1000 are also antiderivatives of $2x$. This fact illustrates one very important property of antiderivatives that can be formulated as follows:

Proposition 4.1

Two antiderivatives of a given function differ only by a constant, i.e. if F and G *are antiderivatives of* f *on* X *then* $F - G = \text{const}$.

Proposition 4.2

If $F(x)$ *is some antiderivative of* $f(x)$ $\forall\, x \in X$, C – *some constant,* $C \in \mathbb{R}$, *then* $F(x)+C$ *is also an antiderivative of function* $f(x)$ *on* X.

Def.: The general form of an antiderivative of f is called ***the indefinite integral of*** f and denoted by

$$\int f(x)dx,$$

where

$f(x)$ is ***the integrand function*** or ***the integrand,***

$f(x)dx$ is ***the integrand***,

x is a ***variable of integration.***

The general form of an antiderivative is $F(x)+C$, where F is an anti-derivative of f and C is an ***arbitrary constant*** or a ***constant of integration,*** and must always be included. So

$$\int f(x)dx = F(x)+C.$$

Thus, for example, $\int 2x\,dx = x^2 + C$.

Remark 4.1

Continuity of f is the ***sufficient condition*** for existence of its antiderivative.

Properties of indefinite integrals

1. $\left(\int f(x)dx\right)' = f(x)$, or $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$.

2. $d\left(\int f(x)dx\right) = f(x)dx$.

3. $\int d\left(F(x)\right) = F(x) + C$, or $\int F'(x)dx = \int d(F(x)) = F(x) + C$.

4. $\forall a \in R,\ a \neq 0\quad \int a\, f(x)dx = a\int f(x)dx.$

5. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx.$

6. If $\int f(x)dx = F(x) + C$, then for any numbers a , b , $a \neq 0$

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C\ .$$

Techniques of integration

To take different types of integrals it is useful to make a list of basic integral formulas (standard integrals) by inverting formulas for derivatives.

Standard integrals

1. $\int 0 \cdot dx = C$
2. $\int 1 \cdot dx = \int dx = x + C$
3. $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C\left(\alpha \neq -1\right)$
4. $\int \frac{1}{x} dx = \ln|x| + C$
5. $\int a^x dx = \frac{a^x}{\ln a} + C$, $\int e^x dx = e^x + C$
6. $\int \sin x dx = -\cos x + C$
7. $\int \cos x dx = \sin x + C$
8. $\int \frac{1}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + C = \ln|\mathrm{cosec}x - \cot x| + C$, $\left(\mathrm{cosec}x = \frac{1}{\sin x}\right)$
9. $\int \frac{1}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C = \ln|\tan x - \sec x| + C$, $\left(\sec x = \frac{1}{\cos x}\right)$

10. $\int \frac{1}{\sin^2 x} dx = -\cot x + C$

11. $\int \frac{1}{\cos^2 x} dx = \tan x + C$

12. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + C(a \neq 0)$

13. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C(a \neq 0)$

14. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + C, |x| < |a|$

15. $\int \frac{dx}{\sqrt{x^2 + a}} = \ln\left|x + \sqrt{x^2 + a}\right| + C, (a \neq 0)$

16. $\int \text{sh}x \cdot dx = \text{ch}x + C \quad \left(\text{sh}x = \frac{e^x - e^{-x}}{2}\right)$

17. $\int \text{ch}x \cdot dx = \text{sh}x + C \quad \left(\text{ch}x = \frac{e^x + e^{-x}}{2}\right)$

Remark 4.2

It is necessary to say a few words about elementary functions which antiderivatives aren't elementary functions: $\int e^{-x^2} dx$, $\int \sin x^2 dx$, $\int \cos x^2 dx$, $\int \frac{e^x}{x} dx$, $\int \frac{\sin x}{x} dx$, $\int \frac{\cos x}{x} dx$, $\int \frac{dx}{\ln x}$, i.e. there is no elementary function $F(x)$, that satisfies the conditions $F'(x) = \sin x^2$ or $F'(x) = \frac{\cos x}{x}$. In this case antiderivatives can be found by means of power series (non-elementary functions).

Example 4.1. Find $\int (5\sin x + 2\cos x - \frac{7}{x} + 3\sqrt{x}) dx$.

□ Applying properties 4, 5 and standard integral 3, we get

$$\int (5\sin x + 2\cos x - \frac{7}{x} + 3\sqrt{x}) dx = 5\int \sin x dx + 2\int \cos x dx - 7\int \frac{dx}{x} + 3\int x^{\frac{1}{2}} dx =$$

$$= -5\cos x + 2\sin x - 7\ln|x| + 3 \cdot \frac{1}{\frac{1}{2}+1} \cdot x^{\frac{3}{2}} + C = -5\cos x + 2\sin x - 7\ln|x| + 2\sqrt[3]{x^2} + C. \blacksquare$$

Example 4.2. Find $\int \cos\left(3x + \frac{\pi}{7}\right) dx$.

□ According to property 6 with $a = 3,\ b = \frac{\pi}{7}$, we have

$$\int \cos\left(3x+\frac{\pi}{7}\right)dx = \;=\frac{1}{3}\sin\left(3x+\frac{\pi}{7}\right)+C. \ \blacksquare$$

Example 4.3. Find $\int \frac{\sin x+2}{\sin^2 x}dx.$

□ Putting each term in the numerator over the denominator with simplifying afterward and applying properties 4,5, it follows, that

$$\int \frac{\sin x+2}{\sin^2 x}dx = \int \frac{1}{\sin x}dx + 2\int \frac{1}{\sin^2 x}dx = \;= \ln\left|\tan\frac{x}{2}\right| + 2\cdot(-\cot x)+C =$$

$$= \ln\left|\tan\frac{x}{2}\right| - 2\cot x + C. \ \blacksquare$$

Example 4.4. Find $\int \frac{x^2}{x^2+4}dx.$

□ Rewrite the integrand in the form: $\frac{x^2}{x^2+4} = \frac{x^2+4-4}{x^2+4} = 1-\frac{4}{x^2+4}$ Then,

$$\int \frac{x^2}{x^2+4}dx = \int 1\cdot dx - 4\int \frac{1}{x^2+4}dx = x - 4\cdot\frac{1}{2}\arctan\frac{x}{2}+C \;= x - 2\arctan\frac{x}{2}+C. \ \blacksquare$$

Example 4.5. Find $\int 5^x\cdot 2^x dx$.

□ Rewrite the integrand function as follows: $5^x\cdot 2^x = (5\cdot 2)^x = 10^x$. Thus,

$$\int 5^x\cdot 2^x dx = \int 10^x dx = \frac{10^x}{\ln 10}+C. \ \blacksquare$$

Example 4.6. Find $\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}dx$.

□ The radicand $4-x^4$ in the denominator can be expressed as $4-x^4=(2+x^2)\cdot(2-x^2)$. Then,

$$\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{2+x^2}\cdot\sqrt{2-x^2}} = \frac{1}{\sqrt{2-x^2}} - \frac{1}{\sqrt{2+x^2}}.$$

$$\int \frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{\sqrt{4-x^4}}dx = \int \frac{1}{\sqrt{\left(\sqrt{2}\right)^2-x^2}}dx - \int \frac{1}{\sqrt{x^2+2}}dx = \arcsin\frac{x}{\sqrt{2}} -$$

$$-\ln\left|x+\sqrt{x^2+2}\right|+C. \ \blacksquare$$

Example 4.7. Find $\int\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right)dx$.

□ Using the difference of squares formula and the trigonometric identity, we get

$$\cos^4\frac{x}{2}-\sin^4\frac{x}{2}=\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)=\cos\left(2\cdot\frac{x}{2}\right)\cdot 1=\cos x.$$

Hence, $\int\left(\cos^4\frac{x}{2}-\sin^4\frac{x}{2}\right)dx=\int\cos x dx=\sin x+C.$ ■

Exercises

1. Find $\int(3-2x)^2\,dx$.
2. Find $\int\frac{1}{x^2+4}dx$.
3. Find $\int\frac{1}{x^3\sqrt{x}}dx$.
4. Find $\int\left(2+\frac{3}{\sin x}\right)^2 dx$.
5. Find $\int 2^{2x}3^x dx$.
6. Find $\int\frac{3x^3+5^x x^5-5x^4}{x^5}dx$.
7. Find $\int\frac{1-\sqrt{4+4x^2}}{4x^2+4}dx$.
8. Find $\int\frac{\sqrt{1+x^2}+2\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$.
9. Find $\int\frac{1-\sin^3 x}{\sin^2 x}dx$.
10. Find $\int\cos^2\frac{x}{2}dx$.

Integration by substitution

Let x be a function of $t: x=\varphi(t)$, where $\varphi(t)$ has a continuous derivative $\varphi'(t)$ and $\varphi(\cdot)$ is a one-to-one correspondence. Then

$$\int f(x)dx=\int f(\varphi(t))\varphi'(t)dt. \qquad (4.1)$$

This formula gives the rule for ***integration by substitution.***

Remark 4.3

Sometimes using the substitution $t=\psi(x)$ is more preferable than $x=\varphi(t)$. In this case the formula (4.1) converts into

$$\int f(\psi(x))\psi'(x)dx=\int f(t)dt.$$

Remark 4.4

Though there is no unified approach to choose an appropriate substitution, it is possible to give some tips:

1. If the integrand involves a composite function $f(\psi(x))$, then the substitution $t=\psi(x)$ is used as a rule.

Example 4.8. Find $\int\frac{\cos\sqrt{x}}{\sqrt{x}}dx$.

□ There is the composite function $f(\psi(x))=\cos\sqrt{x}$ under the integral sign. So the substitution $t=\sqrt{x}$ is appropriate.

$$\int\cos\sqrt{x}\frac{dx}{\sqrt{x}}=\left|\begin{array}{l} t=\sqrt{x} \\ dt=\frac{dx}{2\sqrt{x}} \\ \frac{dx}{\sqrt{x}}=2dt \end{array}\right|=2\int\cos t\,dt=2\sin t+C=2\sin\sqrt{x}+C. \blacksquare$$

2. If the integrand contains the expression $\psi'(x)dx$ that is the differential of $\psi(x)$ then the substitution $t=\psi(x)$ is expedient to use.

Example 4.9. Find $\int\sin^3 x\cdot\cos x\,dx$.

□ The integrand $\sin^3 x\cdot\cos x\,dx$ involves the factor $\cos x\,dx$ that is $\cos x\,dx=(\sin x)'\,dx=d\sin x$. So the substitution $t=\sin x$ can be chosen.

$$\int\sin^3 x\cdot\cos x\,dx=\left|\begin{array}{l} t=\sin x \\ dt=\cos x \end{array}\right|=\int t^3dt=\frac{t^4}{4}+C=\frac{\sin^4 x}{4}+C. \blacksquare$$

Example 4.10. Find $\int\frac{\sqrt[5]{\ln x+2}}{x}dx$.

□ The integrand contains the expression $\frac{dx}{x}$ that can be considered as $d\ln x$ or $d(\ln x+2)$. So either the substitution $t=\ln x$ or $t=\ln x+2$ are possible.

$$\int\sqrt[5]{\ln x+2}\frac{dx}{x}=\left|\begin{array}{l} t=\ln x+2 \\ dt=\frac{dx}{x} \end{array}\right|=\int t^{1/5}dt=\frac{t^{6/5}}{6/5}+C=\frac{5}{6}\sqrt[5]{(\ln x+2)^6}+C. \blacksquare$$

Example 4.11. Find $\int \frac{\sin 3x}{3+\cos 3x}dx$.

□ Since $d(3+\cos 3x) = -3\cdot \sin 3x dx$, let $t = 3+\cos 3x$. Then

$$\int \frac{\sin 3x}{3+\cos 3x}dx = \left| \begin{array}{l} t = 3+\cos 3x \\ dt = -3d\sin 3x \\ d\sin 3x = -\frac{dt}{3} \end{array} \right| = -\frac{1}{3}\int \frac{dt}{t} = -\frac{\ln|t|}{3} + C = -\frac{\ln|3+\cos 3x|}{3} + C. ■$$

Remark 4.5.

It is worthwhile to say that it is possible not to use a new variable of integration explicitly.

Example 4.12. Find $\int e^{3x+4}dx$.

□ Note, that $d(3x+4) = 3dx$, so to get the expression converted into $d(3x+4)$ it's enough to multiply the given integral and then divide it by 3:

$$\int e^{3x+4}dx = \frac{1}{3}\int e^{3x+4}\cdot 3dx = \frac{1}{3}\int e^{3x+4}d(3x+4) = \frac{e^{3x+4}}{3} + C. ■$$

Trigonometric substitutions

Trigonometric substitutions are effective when the following irrational functions $\sqrt{a^2-x^2}$; $\sqrt{a^2+x^2}$; $\sqrt{x^2-a^2}$ arise under the integral sign. For example, making the substitution $x = a\sin\alpha$ allows us to get rid of the radical as $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\alpha} = \sqrt{a^2(1-\sin^2\alpha)} = a\cos\alpha$.

Appropriate substitutions are listed below (see table 4.1).

Table 4.1

Function	Substitution
$\sqrt{a^2-x^2}$	$x = a\sin\alpha$
$\sqrt{a^2+x^2}$	$x = a\tan\alpha$
$\sqrt{x^2-a^2}$	$x = \frac{a}{\cos\alpha} = a\sec\alpha$

Example 4.13. Find $\int \sqrt{4-x^2}dx$.

$$□ \int \sqrt{4-x^2}dx = \left| \begin{array}{l} x = 2\sin\alpha \\ dx = 2\cos\alpha d\alpha \\ \sqrt{4-x^2} = 2\cos\alpha \end{array} \right| = \int 2\cos\alpha\cdot 2\cos\alpha d\alpha = 4\int \cos^2\alpha d\alpha =$$

$$=4\int\frac{1+\cos 2\alpha}{2}d\alpha=2\int d\alpha+2\int\cos 2\alpha d\alpha=2\alpha+\int\cos 2\alpha\, d(2\alpha)=2\alpha+\sin 2\alpha+C=$$

$$=2\alpha+2\sin\alpha\cos\alpha+C=2\arcsin\frac{x}{2}+x\sqrt{1-\frac{x^2}{4}}+C\,.\blacksquare$$

Remark 4.6

It should be pointed out that making a substitution, e.g. $x=a\sin\alpha$, restrictions for α are imposed to get a one-to-one correspondence between x and α: $\alpha\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

Illustrate some other variants of substitutions.

Example 4.14. Find $\int\frac{dx}{x\sqrt{2x+1}}$.

□ $$\int\frac{dx}{x\sqrt{2x+1}}=\left|\begin{array}{l} t=\sqrt{2x+1}\Rightarrow x=\dfrac{t^2-1}{2} \\ dt=\dfrac{dx}{\sqrt{2x+1}}\end{array}\right|=\int\frac{2dt}{t^2-1}=$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=\ln\left|\frac{\sqrt{2x+1}-1}{\sqrt{2x+1}+1}\right|+C.\blacksquare$$

Example 4.15. Find $\int\frac{dx}{e^x+1}$.

□ $$\int\frac{dx}{e^x+1}=\left|\begin{array}{l} t=e^x+1 \\ dt=e^x dx \\ dx=\dfrac{dt}{e^x}=\dfrac{dt}{t-1}\end{array}\right|=\int\frac{dt}{t(t-1)}=\int\frac{((1-t)+t)dt}{t(t-1)}=-\int\frac{dt}{t}+\int\frac{dt}{t-1}=$$

$$=-\ln|t|+\int\frac{d(t-1)}{t-1}=-\ln|t|+\ln|t-1|+C=\ln\left|1-\frac{1}{e^x+1}\right|+C\,.\blacksquare$$

Integration by parts

Let $u(x)$ and $v(x)$ be continuously differentiable functions that means $u'(x)$ and $v'(x)$ exist and $u'(x)$, $v'(x)$ are themselves continuous functions. Then,

$$\big(u(x)v(x)\big)'=u'(x)v(x)+u(x)v'(x).$$

Integrating both sides with respect to x, we get

$$\int (u(x)v(x))' dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx.$$

Taking into consideration that $\int (u(x)v(x))' dx$ is $u(x)v(x)$ up to a constant and $du(x) = u'(x)dx$, we have $u(x)v(x) = \int v(x)du(x) + \int u(x)dv(x)$ or shortly

$$uv = \int v\,du + \int u\,dv$$

and finally

$$\int u\,dv = uv - \int v\,du. \qquad (4.2)$$

We derived the formula for ***integration by parts.*** This formula expresses one integral $\int u\,dv$ in terms of another integral $\int v\,du$. Making a proper choice of u and v the second integral may be easier to evaluate than the first one.

The strategy of calculation includes the following steps. A given integrand is represented as a product of two functions, one of which is taken for u and the other one is chosen as dv. Then we find $du = u'dx$ and $v = \int dv$. We should set a constant of integration equal to zero. At last, we substitute the result in the right-hand side of (4.2) and so complete the routine.

Below there are listed functions which must always be integrated by parts. Also there is given the proper choice of u:

1) $\int P_n(x)\arcsin x\,dx$: $u = \arcsin x$, $dv = P_n(x)dx$, where $P_n(x)$ is the nth degree polynomial.

2) $\int P_n(x)\arccos x\,dx$: $u = \arccos x$, $dv = P_n(x)dx$.

3) $\int P_n(x)\arctan x\,dx$: $u = \arctan x$, $dv = P_n(x)dx$.

4) $\int P_n(x)\operatorname{arccot} x\,dx$: $u = \operatorname{arccot} x$, $dv = P_n(x)dx$,

5) $\int P_n(x)\ln(x)dx$: $u = \ln(x)$, $dv = P_n(x)dx$.

6) $\int x^\alpha \ln(x)dx$: for $\alpha \in R$, $\alpha \neq -1$ $u = \ln(x)$, $dv = x^\alpha dx$.

---

7) $\int P_n(x)e^{\alpha x}dx$: $u = P_n(x)$, $dv = e^{\alpha x}dx$, $\alpha \in R$, $\alpha \neq 0$.

8) $\int P_n(x)\cos\alpha x\,dx$: $u = P_n(x)$, $dv = \cos\alpha x\,dx$, $\alpha \in R$, $\alpha \neq 0$.

9) $\int P_n(x)\sin\alpha x\,dx$: $u = P_n(x)$, $dv = \sin\alpha x\,dx$, $\alpha \in R$, $\alpha \neq 0$.

10) $\int P_n(x)a^x dx$: $u = P_n(x)$, $dv = a^x dx$, $a > 0, a \neq 1$.

11) $\int e^{\alpha x}\cdot\cos bx\,dx$ и $\int e^{\alpha x}\cdot\sin bx\,dx$, $a, b \in R$, $a \neq 0$, $b \neq 0$, either $e^{\alpha x}$ or a trigonometric function may be chosen as u. In this case integration by parts is applied twice. As a result we will receive an equation for the given integral.

Example 4.16. Find $\int(2x+3)\cos 5x dx$.

□ $\int(2x+3)\cos 5x dx$ is a sort of the integral in case 8 with $P_1(x)=2x+3$, $\cos\alpha x=\cos 5x$. So, we take $2x+3$ for u and $\cos 5x dx$ for dv. Then

$$\int(2x+3)\cos 5x dx=\left|\begin{matrix} u=2x+3 & \Rightarrow du=2dx \\ dv=\cos 5x dx \Rightarrow & v=\int\cos 5x dx=\frac{1}{5}\sin 5x\end{matrix}\right|=$$

$$=\frac{2x+3}{5}\sin 5x-\frac{2}{5}\int\sin 5x dx=\frac{2x+3}{5}\sin 5x-\frac{2}{5}\cdot\frac{1}{5}(-\cos 5x)+C=\frac{2x+3}{5}\sin 5x+\frac{2}{25}\cos 5x+C.$$

To evaluate the integrals $\int\cos 5x dx$ and $\int\sin 5x dx$ we can apply property 6 or the substitution $u=5x$. ■

Example 4.17. Find $\int(36x^5+1)\ln x dx$.

□ We deal with case 5, so

$$\int(36x^5+1)\ln x dx=\left|\begin{matrix} u=\ln x & \Rightarrow & du=\frac{dx}{x} \\ dv=(36x^5+1)dx & \Rightarrow & v=\int(36x^5+1)dx=6x^6+x\end{matrix}\right|=$$

$$=\ln x\cdot(6x^6+x)-\int(6x^6+x)\frac{1}{x}dx=\ln x\cdot(6x^6+x)-\int(6x^5+1)dx=$$

$$=\ln x\cdot(6x^6+x)-x^6-x+C. \ ■$$

Example 4.18. Find $\int(x-1)\sin\frac{x}{2}dx$.

□ According to case 9, we have

$$\int(x-1)\sin\frac{x}{2}dx=\left|\begin{matrix} u=x-1 & \Rightarrow du=dx \\ dv=\sin\frac{x}{2}dx\Rightarrow & v=\int\sin\frac{x}{2}dx=-2\cos\frac{x}{2}\end{matrix}\right|=$$

$$=(x-1)\left(-2\cos\frac{x}{2}\right)-\int\left(-2\cos\frac{x}{2}\right)dx=(2-2x)\cos\frac{x}{2}+2\int\cos\frac{x}{2}dx=$$

$$=(2-2x)\cos\frac{x}{2}+2\cdot 2\sin\frac{x}{2}+C=(2-2x)\cos\frac{x}{2}+4\sin\frac{x}{2}+C. \ ■$$

Example 4.19. Find $\int\sqrt{x}\cdot\ln x dx$.

□ The integrand $\sqrt{x}\cdot\ln x$ corresponds to case 6. Thus,

$$\int \sqrt{x}\cdot \ln x dx = \left| \begin{matrix} u = \ln x & \Rightarrow & du = \dfrac{dx}{x} \\ dv = \sqrt{x}dx & \Rightarrow & v = \int \sqrt{x}dx = \int x^{1/2}dx = 2\dfrac{x^{3/2}}{3} \end{matrix} \right| =$$

$$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int \frac{x^{3/2}}{x}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C = \frac{2}{3}x^{3/2}\left(\ln x - \frac{2}{3} \right) + C. \blacksquare$$

Example 4.20. Find $\int \left(x^2+3\right)e^x dx$.

□ According to case 7, we have

$$\int \left(x^2+3\right)e^x dx = \left| \begin{matrix} u = x^2+3 & \Rightarrow & du = 2xdx \\ dv = e^x dx & \Rightarrow & v = \int e^x dx = e^x \end{matrix} \right| = \left(x^2+3\right)e^x - \int e^x \cdot 2xdx =$$

The result involves the integral $\int e^x \cdot 2xdx$ that is taken by integration by parts as well. So, sometimes we have to use integration by parts more than once.

$$= \left| \begin{matrix} u = 2x & \Rightarrow & du = 2dx \\ dv = e^x dx & \Rightarrow & v = \int e^x dx = e^x \end{matrix} \right| = \left(x^2+3\right)e^x - \left(2x\cdot e^x - \int e^x \cdot 2dx\right) =$$

$$= \left(x^2+3\right)e^x - 2x\cdot e^x + 2e^x + C = \left(x^2-2x+5\right)e^x + C \blacksquare$$

Example 4.21. Find $\int (2x+1)\cdot \arctan x\, dx$.

□ We deal with case 3, so

$$\int (2x+1)\cdot \arctan x dx = \left| \begin{matrix} u = \arctan x & \Rightarrow & du = \dfrac{dx}{1+x^2} \\ dv = (2x+1)dx & \Rightarrow & v = \int (2x+1)dx = x^2+x \end{matrix} \right| =$$

$$= \arctan x\cdot\left(x^2+x\right) - \int \left(x^2+x\right)\cdot \frac{1}{1+x^2}dx = \left| \begin{matrix} \int \dfrac{x^2+x}{x^2+1} = \int \dfrac{\left(x^2+1\right)-1+x}{x^2+1} = \\ = \int \left(1 - \dfrac{1}{1+x^2} + \dfrac{x}{x^2+1}\right)dx = \\ = \left| \begin{matrix} t = x^2+1 \\ dt = 2xdx \\ xdx = \dfrac{dt}{2} \end{matrix} \right| = x - \arctan x + \dfrac{1}{2}\int \dfrac{dt}{t} = x - \arctan x + \dfrac{1}{2}\ln|t| = \\ = x - \arctan x + \dfrac{1}{2}\ln\left|x^2+1\right| + C \end{matrix} \right|$$

$$= \left(x^2+x+1\right)\cdot \arctan x - x - \frac{1}{2}\ln\left(x^2+1\right) + C. \blacksquare$$

Example 4.22. Find $\int e^{2x}\cdot\cos x dx$.

□ In this case we are free in a choice of u. Let $u=e^{2x}$.

$$\int e^{2x}\cdot\cos x dx=\left|\begin{matrix} u=e^{2x} & \Rightarrow & du=2e^{2x}dx \\ dv=\cos x dx & \Rightarrow & v=\int\cos x dx=\sin x\end{matrix}\right|=e^{2x}\cdot\sin x-2\int e^{2x}\cdot\sin x dx=$$

$$=\left|\begin{matrix} u=e^{2x} & \Rightarrow & du=2e^{2x}dx \\ dv=\sin x dx & \Rightarrow & v=\int\sin x dx=-\cos x\end{matrix}\right|=e^{2x}\cdot\sin x-2\left(-e^{2x}\cdot\cos x-2\int e^{2x}\cdot(-\cos x)dx\right)=$$

$$=e^{2x}\cdot\sin x+2e^{2x}\cdot\cos x-4\int e^{2x}\cdot\cos x\,dx.$$

Denote $\int e^{2x}\cdot\cos x dx$ as I, then

$$I=e^{2x}\cdot\sin x+2e^{2x}\cdot\cos x-4I\Rightarrow I=\frac{1}{5}e^{2x}\left(\sin x+2\cos x\right)+C.\ ■$$

Exercises

1. Find $\int 2^{1-5x}dx$
2. Find $\int\frac{3x^2}{1+x^6}dx$
3. Find $\int\frac{3^{\sqrt{x}}}{2\sqrt{x}}dx$
4. Find $\int\sqrt{1+x^2}dx$
5. Find $\int\sqrt{x^2-4}dx$
6. Find $\int\frac{\sin(\ln x)}{x}dx$
7. Find $\int\frac{\ln x}{x^2}dx$
8. Find $\int x\cdot 3^x dx$
9. Find $\int\ln^2 dx$
10. Find $\int x^2\cos 2x dx$
11. Find $\int\frac{(m+nx)\sqrt{m\ln x+nx}}{x}dx$, if *m* is a student's number, *n* is the last numeral in a group number
12. Find $\int\frac{2mxdx}{(mx^2+n)^m}$, if *m* is a student's number, *n* is the last numeral in a group number
13. Find $\int(x+m)\sin nx dx$, if *m* is a student's number, *n* is the last numeral in a group number
14. Find $\int(mx+n)\arctan nx dx$, if *m* is a student's number, *n* is the last numeral in a group number

Integration of rational functions

Let $P_n(x)=a_0x^n+a_1x^{n-1}+...+a_{n-1}x+a_n$ and $Q_m(x)=b_0x^m+b_1x^{m-1}+...+$ $+b_{m-1}x+b_m$ be the n-th degree and the m-th degree polynomials with real coefficients respectively.

Theorem 4.1 (The Fundamental Theorem of Algebra)

There are several versions of the theorem. The first one sounds as:

A polynomial with complex coefficients has at least one zero in the set of complex numbers.

The second version states that

A n-th degree polynomial with complex coefficients has exactly n *zeros in the set of complex numbers counting repeated roots.*

Remark 4.7

The set of real numbers $\mathbb{R}$ is a subset of the set of complex numbers $\mathbb{C}$. Indeed, any real number x can be represented as $x+iy,\ y=0$. Thus, the theorem holds for polynomials with real coefficients as well.

Example 4.23. Factorize the polynomial $P_3(x)=x^3-x^2-9x+9$.

□ Grouping the first two terms together and the last two terms together than factoring the greatest common factor out of each group, we have

$$P_3(x)=x^3-x^2-9x+9=(x^3-x^2)-(9x-9)=x^2(x-1)-9(x-1)=$$
$$=(x-1)(x-3)(x+3).$$

Using the obtained factorization the multiplicity of each root can be determined as a power of a corresponding factor. For example, the factor $(x-1)$ is raised to the first power, so the given polynomial $P_3(x)$ has a root 1 of multiplicity 1. Such a root is called a ***simple root***. If the multiplicity of a root is greater than 1, the root is called a ***multiple root*** or ***repeated root***.

Generally, the multiplicity can be defined as follows. Suppose we have a polynomial $P(x)$ with a root $x=a$ that means $P(a)=0$. And the kth derivative of $P(x)$ differs from zero at $x=a$ while its derivatives of order less than k are zero: $P'(a)=0, P''(a)=0,..., P^{(k-1)}(a)=0, P^{(k)}(a)\neq 0$. Then the multiplicity of the root $x=a$ is k. Thus, $P_3(x)$ has 3 simple roots: -3, 1 and 3.■

Example 4.24. Factorize the polynomial $P_3(x)=x^3-2x^2+9x-18$.

□ Following the strategy from the previous example, we get

$$P_3(x)=x^3-2x^2+9x-18=(x^3-2x^2)+(9x-18)=x^2(x-2)+9(x-2)=$$

$=(x-2)(x-3i)(x+3i)$. So $P_3(x)$ has one real root 2 and two complex roots $-3i, 3i$. ■

Remark 4.8

If a polynomial has a complex root $a+ib$, its complex conjugate $a-ib$ is also a root of the polynomial.

Theorem 4.2

If a polynomial is identically equal to zero, then the all of its coefficients are zero.

Theorem 4.3

If two polynomials are identical to each other then coefficients of one of them are equal to the corresponding coefficients of the other one.

Suppose $P_3(x) = ax^3 + bx^2 + cx + d$, $Q_3(x) = x^3 - 3x^2 + x - 1$. Then $P_3(x) \equiv Q_3(x)$ implies that $a = 1, b = -3, c = 1, d = -1$.

Def.: A function $R(x) = \dfrac{P_n(x)}{Q_m(x)}$ is called a ***rational function*** or ***rational fraction***.

Def.: $R(x)$ is a ***proper fraction*** if $n < m$. Otherwise for $n \geq m$ $R(x)$ is an ***improper fraction.***

Theorem 4.4

Any rational function $\dfrac{P_n(x)}{Q_m(x)}$ *can be written as a sum of a polynomial and a proper rational fraction, i.e.*

$$\frac{P_n(x)}{Q_m(x)} = L_{n-m}(x) + \frac{S(x)}{Q_m(x)}, \tag{4.3}$$

where $L_{n-m}(x)$ *is a polynomial of degree n–m,* $\dfrac{S(x)}{Q_m(x)}$ *is a proper fraction.*

The representation (4.3) can be obtained by means of carrying out the long division.

Example 4.25. Express $\dfrac{2x^3 + x^2 - 9}{x^2 + 2x + 3}$ in the form (4.3).

□ The numerator $2x^3 + x^2 - 9$ is a polynomial of degree 3 ($n = 3$), the denominator $x^2 + 2x + 3$ is a polynomial of degree 2 ($m = 2$), i.e. $n > m$, that means $\dfrac{2x^3 + x^2 - 9}{x^2 + 2x + 3}$ is an improper fraction. Carry out the long division:

$$\begin{array}{r|l} 2x^3 + x^2 \qquad\quad - 9 & x^2 + 2x + 3 \\ \underline{2x^3 + 4x^2 + 6x \qquad} & 2x - 3 \\ -3x^2 - 6x - 9 & \\ -3x^2 - 6x - 9 & \\ 0 & \end{array}$$

Thus, $\dfrac{2x^3 + x^2 - 9}{x^2 + 2x + 3} = 2x - 3$. Comparing the result with (4.3) $L_1(x) = 2x - 3$. ■

Example 4.26. Express $\frac{2x^3+x^2-9}{x-1}$ in the form (4.3).

□ The fraction $\frac{2x^3+x^2-9}{x-1}$ is an improper fraction for the same reason, $n=3>m=1$. Carry out the long division:

$$\begin{array}{r|l} 2x^3+x^2 \qquad -9 & x-1 \\ \underline{2x^3-2x^2} \qquad\quad & 2x^2+3x+3 \\ 3x^2 \qquad -9 & \\ \underline{3x^2-3x} \qquad & \\ 3x-9 & \\ \underline{3x-3} & \\ -6 & \end{array}$$

Hence, $\frac{2x^3+x^2-9}{x-1}=2x^2+3x+3-\frac{6}{x-1}$. In the case $L_2(x)=2x^2+3x+3$, $\frac{S(x)}{Q_1(x)}=-\frac{6}{x-1}$. ■

Def.: x^2+px+q is ***irreducible***, if the corresponding quadratic equation $x^2+px+q=0$ has no real solutions. In this case the discriminant $D=p^2-4q<0$.

Def.: Proper fractions of form $\frac{A}{(x-\alpha)^k}$ or $\frac{Mx+N}{(x^2+px+q)^l}$ are called ***partial fractions***, if $k,l\in N$, x^2+px+q is irreducible

Theorem 4.5

Any proper fraction $\frac{S(x)}{Q_m(x)}$ *whose denominator* $Q_m(x)$ *has the form*

$$Q_m(x)=b_0\overbrace{(x-\alpha_1)^{k_1}\ldots(x-\alpha_r)^{k_r}}^{\text{linear factors}}\cdot\overbrace{\underbrace{(x^2+p_1x+q_1)^{l_1}}_{D_1<0}\ldots\underbrace{(x^2+p_sx+q_s)^{l_s}}_{D_s<0}}^{\text{quadratic irreducible factors}},$$

$$k_1+\ldots+k_r+2(l_1+\ldots+l_s)=m,$$

can be expressed as a finite sum of partial fractions:

$$\frac{S(x)}{Q_m(x)}=R_1+R_2+\ldots+R_s, \qquad (4.4)$$

where $R_i, i=1,..,s$ *is a partial fraction of the form given in the definition.*

The representation (4.4) is called the ***partial fraction decomposition***. The decomposition (4.4) can be rewritten in the following expanded form:

$$\frac{S(x)}{Q_m(x)} = \underbrace{\frac{A_1}{x-\alpha_1} + \frac{A_2}{(x-\alpha_1)^2} + \ldots + \frac{A_{k_1}}{(x-\alpha_1)^{k_1}}}_{k_1 \text{ items}} + \ldots + \underbrace{\frac{B_1}{x-\alpha_r} + \ldots + \frac{B_{k_r}}{(x-\alpha_r)^{k_r}}}_{k_r \text{ items}} + \ldots$$

$$+ \underbrace{\frac{M_1x+N_1}{x^2+p_1x+q_1} + \frac{M_2x+N_2}{(x^2+p_1x+q_1)^2} + \ldots + \frac{M_{l1}x+N_{l1}}{(x^2+p_1x+q_1)^{l_1}}}_{l_1 \text{ items}} + \ldots +$$

$$+ \underbrace{\frac{K_1x+L_1}{x^2+p_sx+q_s} + \ldots + \frac{K_{ls}x+N_{ls}}{(x^2+p_sx+q_s)^{l_s}}}_{l_s \text{ items}}.$$

Remark 4.9

1. All numerator's coefficients are undetermined real numbers that are needed to find.
2. If a given function is an improper fraction then the long division should be employed to reduce the problem to integration of a proper fraction.

Integration of partial fractions

1. $\int \frac{A}{x-\alpha}dx = \left| \begin{matrix} u = x-a \\ du = dx \end{matrix} \right| = A\int \frac{du}{u} = A\ln|u| + C = A \cdot \ln|x-\alpha| + C;.$

2. $\int \frac{A}{(x-\alpha)^k}dx,\ k>1$: $\int \frac{A}{(x-\alpha)^k}dx = A \cdot \int (x-\alpha)^{-k}dx = \left| \begin{matrix} u = x-a \\ du = dx \end{matrix} \right| =$

$$= A\int u^{-k}du = A\frac{u^{-k+1}}{-k+1} = A\frac{(x-\alpha)^{-k+1}}{-k+1} + C = \frac{A}{-k+1} \cdot \frac{1}{(x-\alpha)^{k-1}} + C.$$

3. $\int \frac{Mx+N}{x^2+px+q}dx =$

$$\left| \begin{array}{l} u = x + \frac{p}{2},\ \ du = dx,\ \ x = u - \frac{p}{2}. \\ \text{Complete the square: } x^2+px+q = \left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right) = u^2 + a^2, \\ \text{where } a^2 = q - \frac{p^2}{4} > 0. \end{array} \right| =$$

$$\int \frac{M\left(u-\frac{p}{2}\right)+N}{u^2+a^2}du = M\int \frac{u}{u^2+a^2}du + \int \frac{-\frac{Mp}{2}+N}{u^2+a^2}du =$$

$$= \frac{M}{2a}\ln\left(u^2+a^2\right) + \frac{2N-Mp}{2a}\arctan\frac{u}{a} + C.$$

The first integral is found by use of the substitution $t = u^2 + a^2$. The second is a standard integral. To accomplish integrating we should make the reverse substitution $u = x + \frac{p}{2}$:

$$\int \frac{Mx+N}{x^2+px+q}dx = \frac{M}{\sqrt{4q-p^2}}\ln\left(x^2+x+1\right) + \frac{2N-Mp}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C.$$

4. $\int \frac{Mx+N}{\left(x^2+px+q\right)^k}dx \quad k>1,\ D<0$. Using the substitution $u = x + \frac{p}{2}$ the integral can be converted into a sum:

$$\int \frac{Mx+N}{\left(x^2+px+q\right)^k}dx = M\int \frac{udu}{\left(u^2+a^2\right)^k} + \left(N - \frac{Mp}{2}\right)\int \frac{du}{\left(u^2+a^2\right)^k},\quad a^2 = q - \frac{p^2}{4}.$$

The first integral can be easily calculated by use of the substitution $t = u^2 + a^2$. To take the second one we should rewrite it as follows:

$$\int \frac{du}{\left(u^2+a^2\right)^k} = \frac{1}{a^2}\int \frac{\left(u^2+a^2\right)-u^2}{\left(u^2+a^2\right)^k}du = \frac{1}{a^2}\left(\int \frac{du}{\left(u^2+a^2\right)^{k-1}} - \int \frac{u^2du}{\left(u^2+a^2\right)^k}\right).$$

If we denote $\int \frac{du}{\left(u^2+a^2\right)^k}$ by I_k then we will get the recurrent formula:

$$I_k = \frac{1}{a^2}\left(I_{k-1} - \int \frac{u^2du}{\left(u^2+a^2\right)^k}\right). \qquad (4.5)$$

Example 4.27. Find $\int \frac{6dx}{2x+3}$.

□ The function $\frac{6}{2x+3}$ is a partial fraction (case 1). Then

$$\int \frac{6dx}{2x+3} = \left| \begin{array}{l} u = 2x+3 \\ du = 2dx \\ dx = \frac{du}{2} \end{array} \right| = \frac{6}{2}\int \frac{du}{u} = 3\ln|u| + C = 3\ln|x+1,5| + C. ■$$

Example 4.28. Find $\int \frac{dx}{x^3+6x^2+12x+8}$.

□ The function $\frac{1}{x^3+6x^2+12x+8}$ can be rewritten as

$$\frac{1}{x^3+6x^2+12x+8}=\frac{1}{(x+2)^3}.$$

So we deal with a partial fraction (case 2). Then

$$\int \frac{dx}{x^3+6x^2+12x+8}=\int \frac{dx}{(x+2)^3}=\left|u=x+2\right|=\int u^{-3}du=\frac{u^{-3+1}}{-3+1}+C=-\frac{1}{2(x+2)^2}+C. \blacksquare$$

Example 4.29. Find $\int \frac{2x+1}{x^2+x+1}dx$.

□ The function $\frac{2x+1}{x^2+x+1}$ is a partial fraction (case 3), because $2x+1$ is a linear function, x^2+x+1 is a quadratic irreducible function with the negative discriminant $D=1^2-4=-3$.

Two different ways of solving the problem are demonstrated below

I. Completing the square in the denominator we have

$$x^2+x+1=\left\{x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\right\}+1-\left(\frac{1}{2}\right)^2=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

Then $\int \frac{2x+1}{x^2+x+1}dx=\int \frac{2x+1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx=\left|\begin{matrix}u=x+\frac{1}{2}\Rightarrow x=u-\frac{1}{2}\\ du=dx\end{matrix}\right|=$

$$\int \frac{2\left(u-\frac{1}{2}\right)+1}{u^2+\frac{3}{4}}du=\int \frac{2u}{u^2+\frac{3}{4}}du=\left|\begin{matrix}t=u^2+\frac{3}{4}\\ dt=2udu\end{matrix}\right|=\int \frac{dt}{t}=\ln|t|+C=$$

$$\ln\left(u^2+\frac{3}{4}\right)+C=\ln\left|x^2+x+1\right|+C.$$

II. Note: $\left(x^2+x+1\right)'=2x+1\Rightarrow d\left(x^2+x+1\right)=(2x+1)dx$. Then

$$\int \frac{(2x+1)dx}{x^2+x+1}=\int \frac{d\left(x^2+x+1\right)}{x^2+x+1}=\ln\left|x^2+x+1\right|+C.$$

The first way is more general than the second one. ■

Example 4.30. Find $\int \frac{x+5}{\left(x^2+x+1\right)^2}dx.$

□ The function $\frac{x+5}{\left(x^2+x+1\right)^2}$ is a partial fraction (case 4). Complete the square in the denominator first:

$$x^2+x+1=\left\{x^2+2x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2\right\}+1-\left(\frac{1}{2}\right)^2=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}.$$

Then

$$\int\frac{x+5}{\left(x^2+x+1\right)^2}dx=\left|\begin{matrix}u=x+\frac{1}{2}\Rightarrow x=u-\frac{1}{2}\\ du=dx\end{matrix}\right|=\int\frac{u-\frac{1}{2}+5}{\left(u^2+\frac{3}{4}\right)^2}du=$$

$$=\int\frac{u}{\left(u^2+\frac{3}{4}\right)^2}du+\frac{9}{2}\int\frac{1}{\left(u^2+\frac{3}{4}\right)^2}du.$$

To take the first integral we can use the substitution $t=u^2+\frac{3}{4}$ and the formula (4.5) can be applied for the second one.

- $$\int\frac{u}{\left(u^2+\frac{3}{4}\right)^2}du=\left|\begin{matrix}t=u^2+\frac{3}{4}\\ dt=2udu\\ udu=\frac{dt}{2}\end{matrix}\right|=\frac{1}{2}\int\frac{dt}{t^2}=-\frac{1}{2t}+C=-\frac{1}{2\left(u^2+\frac{3}{4}\right)}+C.$$

- $$\frac{9}{2}\int\frac{1}{\left(u^2+\frac{3}{4}\right)^2}du=\left|k=2,\ a^2=\frac{3}{4}\right|=\frac{9}{2}\cdot\frac{4}{3}\cdot\left(\int\frac{du}{u^2+\frac{3}{4}}-\int\frac{u^2du}{\left(u^2+\frac{3}{4}\right)^2}\right)=$$

$$=\left|\begin{matrix}z=u\quad\Rightarrow\quad dz=du\\ dv=\frac{udu}{\left(u^2+\frac{3}{4}\right)^2}\Rightarrow v=\int\frac{udu}{\left(u^2+\frac{3}{4}\right)^2}=\left|\begin{matrix}t=u^2+\frac{3}{4}\\ dt=2udu\\ udu=\frac{dt}{2}\end{matrix}\right|=\\ =\frac{1}{2}\int\frac{dt}{t^2}=-\frac{1}{2\left(u^2+\frac{3}{4}\right)}\end{matrix}\right|=\frac{9}{2}\left(\frac{2}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}+\right.$$

$$\left.+\frac{u}{2\left(u^2+\frac{3}{4}\right)}-\int\frac{du}{2\left(u^2+\frac{3}{4}\right)}\right)=6\left(\frac{2}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}+\frac{u}{2\left(u^2+\frac{3}{4}\right)}-\frac{1}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}\right)+C=$$

$$= 6\left(\frac{1}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}+\frac{u}{2\left(u^2+\frac{3}{4}\right)}\right)+C.$$

Finally, we have

$$\int\frac{x+5}{\left(x^2+x+1\right)^2}dx=\int\frac{u}{\left(u^2+\frac{3}{4}\right)^2}du+\frac{9}{2}\int\frac{1}{\left(u^2+\frac{3}{4}\right)^2}du=-\frac{1}{2\left(u^2+\frac{3}{4}\right)}+$$

$$+6\left(\frac{1}{\sqrt{3}}\arctan\frac{2u}{\sqrt{3}}+\frac{u}{2\left(u^2+\frac{3}{4}\right)}\right)+C=-\frac{1}{2\left(x^2+x+1\right)}+2\sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}+$$

$$+\frac{3}{2}\frac{2x+1}{\left(x^2+x+1\right)}+C.\blacksquare$$

Integration of an arbitrary rational fraction

Algorithm

1. Define whether a given fraction $\frac{P_n(x)}{Q_m(x)}$ is a proper fraction or not. If the given fraction is an improper fraction, it should be represented in the form (4.3) applying theorem 4.4:
$$\frac{P_n(x)}{Q_m(x)}=L_{n-m}(x)+\frac{S(x)}{Q_m(x)}$$
Otherwise, move on to step 2.
2. Factorize the denominator $Q_m(x)$:
$$Q_m(x)=b_0(x-\alpha_1)^{k_1}...(x-\alpha_r)^{k_r}\cdot\left(x^2+p_1x+q_1\right)^{l_1}...\left(x^2+p_sx+q_s\right)^{l_s},$$
$$k_1+...+k_r+2\left(l_1+...+l_s\right)=m,$$
3. Apply theorem 4.5 to carry out the partial fraction decomposition:
$$\frac{S(x)}{Q_m(x)}=R_1+R_2+...+R_s,$$
where $R_i, i=1,..,s$ is a partial fraction of one of the following forms:
$$\frac{A}{\left(x-\alpha\right)^k},\ k=1,...,k_1;\ \frac{Mx+N}{\left(x^2+px+q\right)^l},\ l=1,..,l_1$$
4. Integrate the obtained partial fractions using the results of cases 1-4.

Example 4.31. Find $\int \frac{2x^3+x^2-9}{x-1}dx$.

□ Apply the algorithm to the given fraction $\frac{2x^3+x^2-9}{x-1}$.

1. To reduce the given improper fraction to a proper one we can use the result of example 4.26:

$$\frac{2x^3+x^2-9}{x-1} = 2x^2+3x+3-\frac{6}{x-1}.$$

2.- 3. The obtained proper fraction $\frac{6}{x-1}$ is a partial fraction of an appropriate form.

4. For the final result it's enough to take the integral

$$\int\left(2x^2+3x+3-\frac{6}{x-1}\right)dx = \frac{2}{3}x^3+2x^2+3x-6\ln|x-1|+C. \blacksquare$$

Example 4.32. Find $\int \frac{x^3+x+1}{x\left(x^2+1\right)}dx$.

□ Applying the algorithm we have:

1. To convert the improper fraction $\frac{x^3+x+1}{x\left(x^2+1\right)}$ into a proper one we can group the first term of the numerator with the second one and leave the last term alone, then divide each obtained group by the denominator where we previously remove parenthesis:

$$\frac{x^3+x+1}{x\left(x^2+1\right)} = \frac{\left(x^3+x\right)+1}{x^3+x} = \frac{x^3+x}{x^3+x}+\frac{1}{x^3+x} = 1+\frac{1}{x^3+x}.$$

2. The denominator x^3+x is already factorized as $x\left(x^2+1\right)$.

3. The corresponding partial fraction decomposition has a form:

$$\frac{1}{x\left(x^2+1\right)} = \frac{A}{x}+\frac{Mx+N}{x^2+1} = \frac{Ax^2+A+Mx^2+Nx}{x^3+x},$$

where A, M, N are undetermined coefficients. Since two fractions with identical denominators are equal their numerators must be equal as well. According to theorem 4.3 equality of two polynomials is equivalent to equality of coefficients of like powers of x. Equating such coefficients leads us to the system of equations:

$$\begin{cases} A+M=0, \\ N=0, \\ A=1, \end{cases} \Rightarrow \begin{cases} M=-1, \\ N=0, \\ A=1 \end{cases}.$$

Here we compare the coefficients of x^2, x^1 and x^0 sequentially.

4. Integrating the result we get

$$\int\frac{x^3+x+1}{x\left(x^2+1\right)}dx=\int\left(1+\frac{1}{x}-\frac{x}{x^2+1}\right)dx=x+\ln|x|-\int\frac{x}{x^2+1}dx=\left|\begin{matrix}u=x^2+1\\ du=2xdx\\ xdx=\dfrac{du}{2}\end{matrix}\right|=$$

$$=x+\ln|x|-\frac{1}{2}\int\frac{du}{u}=x+\ln|x|-\frac{1}{2}\ln|u|+C=x+\ln|x|-\frac{1}{2}\ln\left(x^2+1\right)+C=x+\ln\left|\frac{x}{\sqrt{x^2+1}}\right|+C.\blacksquare$$

Example 4.33. Find $\int\frac{x^4+1}{x^3+1}dx.$

□ 1. To transform the improper fraction $\frac{x^4+1}{x^3+1}$ ($n=4>m=3$) to a proper one we employ the long division:

$$\begin{array}{r|l} x^4+1 & x^3+1 \\ \underline{x^4+x} & x \\ -x+1 & \end{array}$$

So $\frac{x^4+1}{x^3+1}=x+\frac{-x+1}{x^3+1}$.

2. The denominator x^3+1 can be factorized as $x^3+1=(x+1)\left(x^2-x+1\right)$.

3. The partial fraction decomposition of $\frac{-x+1}{x^3+1}$ has a form:

$$\frac{-x+1}{x^3+1}=\frac{-x+1}{(x+1)\left(x^2-x+1\right)}=\frac{A}{x+1}+\frac{Mx+N}{x^2-x+1}=$$

$=\frac{A\left(x^2-x+1\right)+(x+1)(Mx+N)}{x^3+1}$. Then using the method of undetermined coefficients described above we form the system

$$\begin{cases}A+M=0,\\ -A+N+M=-1,\\ A+N=1,\end{cases}\Leftrightarrow\begin{cases}M=-A\\ N=1-A\\ -A+(1-A)-A=-1,\end{cases}\Leftrightarrow\begin{cases}M=-A\\ N=1-A\\ -3A=-2,\end{cases}$$

wherefrom $A=\frac{2}{3},M=-\frac{2}{3},N=\frac{1}{3}$. Finally $\frac{x^4+1}{x^3+1}=x+\frac{\frac{2}{3}}{x+1}+\frac{-\frac{2}{3}x+\frac{1}{3}}{x^2-x+1}$.

4. $\int\frac{x^4+1}{x^3+1}dx=\int\left(x+\frac{2}{3}\cdot\frac{1}{x+1}+\frac{1}{3}\cdot\frac{-2x+1}{x^2-x+1}\right)dx=\int xdx+$

$$\frac{2}{3}\int\frac{1}{x+1}dx-\frac{1}{3}\int\frac{2x-1}{x^2-x+1}dx=\left|\begin{matrix}u=x+1\\ du=dx\end{matrix}\right|\cup\left|\begin{matrix}t=x^2-x+1\\ dt=(2x-1)dx\end{matrix}\right|=$$

$$=\frac{x^2}{2}+\frac{2}{3}\int\frac{du}{u}-\frac{1}{3}\int\frac{dt}{t}==\frac{x^2}{2}+\frac{2}{3}\ln|u|-\frac{1}{3}\ln|t|+C=$$

$$=\frac{x^2}{2}+\frac{2}{3}\ln|x+1|-\frac{1}{3}\ln|x^2-x+1|+C.\blacksquare$$

Exercises

1. Find $\int\frac{dx}{x^3-1}$.

2. Find $\int\frac{xdx}{x^2-4x-5}$.

3. Find $\int\frac{(2x-3)dx}{(x-1)(x+2)}$.

4. Find $\int\frac{7dx}{(x+1)^6}$.

5. Find $\int\frac{3dx}{x+{}^3\!/\!_4}$.

6. Find $\int\frac{(2x+1)dx}{(x^2+2x+5)^2}$.

7. Find $\int\frac{dx}{(x^2+1)(x^2+4)}$.

8. Find $\int\frac{5x-3n-3m}{x^2-(m+n)x+mn}dx$, if m is a student's number, n is the last numeral in a group number.

9. Find $\int\frac{x^2-n^2}{x^2-(m+n)x+mn}dx$, if m is a student's number, n is the last numeral in a group number.

Integration of trigonometric expressions

Method of integration is chosen depending on the form of a given integrand function. The methods considered below are based on rationalizing an integrand function. Rationalization is carried out by means of substitution. The following cases can be distinguished:

1. $\int \cos^m x \cdot \sin^n x\,dx$.

a) If $m=2k, n=2l+1, k,l \in \mathbb{N}$, $\cos x$ is taken for a new variable t or $t=\cos x$.

Example 4.34. Find $\int \cos^2 x \cdot \sin^3 x\,dx$.

$$\square \int \cos^2 x \cdot \sin^3 x\,dx = \int \cos^2 x \cdot \sin^2 x \cdot \sin x\,dx = \left| \begin{array}{l} t=\cos x, \\ \sin^2 x = 1-\cos^2 x = 1-t^2 \\ dt = \sin x\,dx \end{array} \right| =$$

$$= \int t^2\left(1-t^2\right)dt = \int \left(t^2-t^4\right)dt = \frac{t^3}{3}-\frac{t^5}{5}+C = \frac{\cos^3 x}{3}-\frac{\cos^5 x}{5}+C. \ \blacksquare$$

b) If $m=2k+1, n=2l, k,l \in \mathbb{N}$, $\sin x$ is taken for a new variable t or $t=\sin x$.

Example 4.35. Find $\int \cos^3 x \cdot \sin^4 x\,dx$.

$$\square \int \cos^3 x \cdot \sin^4 x\,dx = \int \cos^2 x \cdot \sin^4 x \cdot \cos x\,dx = \left| \begin{array}{l} t=\sin x, \\ \cos^2 x = 1-\sin^2 x = 1-t^2 \\ dt = \cos x\,dx \end{array} \right| =$$

$$= \int \left(1-t^2\right)t^4 dt = \int \left(t^4-t^6\right)dt = \frac{t^5}{5}-\frac{t^7}{7}+C = \frac{\cos^5 x}{5}-\frac{\cos^7 x}{7}+C. \blacksquare$$

c) If $m=2k, n=2l, k,l \in \mathbb{N}$, the integral is calculated by use of trigonometric identities:

$$\sin^2 x = \frac{1-\cos 2x}{2}; \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

Example 4.36. Find $\int \sin^2 3x\,dx$.

$$\square \int \sin^2 3x\,dx = \left| \sin^2 3x = \frac{1-\cos 6x}{2} \right| = \int \frac{1-\cos 6x}{2}dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 6x\,dx =$$

$$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{6}\sin 6x + C = \frac{x}{2}-\frac{\sin 6x}{12}+C. \ \blacksquare$$

2. $\int \sin\alpha x\cdot\cos\beta x\,dx,\ \int\cos\alpha x\cdot\cos\beta x\,dx,\ \int\sin\alpha x\cdot\sin\beta x\,dx$.

Using trigonometric formulas allow to reduce the given integral to the integral of a sum or a difference of cosine or sine functions.

$$\sin\alpha x\cdot\cos\beta x=\frac{1}{2}\left[\sin(\alpha+\beta)x+\sin(\alpha-\beta)x\right];$$

$$\sin\alpha x\cdot\sin\beta x=\frac{1}{2}\left[\cos(\alpha-\beta)x-\cos(\alpha+\beta)x\right];$$

$$\cos\alpha x\cdot\cos\beta x=\frac{1}{2}\left[\cos(\alpha-\beta)x+\cos(\alpha+\beta)x\right].$$

Example 4.37. Find $\int\sin 3x\cdot\cos 5x\,dx$.

□ $\int\sin 3x\cdot\cos 5x\,dx=\frac{1}{2}\int\left(\sin(3x+5x)+\sin(3x-5x)\right)dx=\frac{1}{2}\int\left(\sin(8x)-\sin(2x)\right)dx=$

$=\frac{1}{2}\int\sin(8x)dx-\frac{1}{2}\int\sin(2x)dx=-\frac{1}{16}\cos(8x)-\frac{1}{4}\cos(2x)+C.$ ■

Example 4.38. Find $\int\frac{(\sin\frac{x}{2}-\sin^3\frac{x}{2})dx}{\cos\frac{x}{2}}$.

□ $\int\frac{\left(\sin\frac{x}{2}-\sin^3\frac{x}{2}\right)dx}{\cos\frac{x}{2}}=\int\frac{\sin\frac{x}{2}\left(1-\sin^2\frac{x}{2}\right)dx}{\cos\frac{x}{2}}=\left|1-\sin^2\frac{x}{2}=\cos^2\frac{x}{2}\right|=$

$=\int\frac{\sin\frac{x}{2}\cos^2\frac{x}{2}dx}{\cos\frac{x}{2}}=\int\sin\frac{x}{2}\cos\frac{x}{2}dx=\frac{1}{2}\int\sin x\,dx=-\frac{\cos x}{2}+C.$ ■

Example 4.39. Find $\int\sin^2 3x\,dx$.

□ $\int\sin^2 3x\,dx=\left|\sin^2 3x=\frac{1-\cos 6x}{2}\right|=\int\frac{1-\cos 6x}{2}dx=\frac{1}{2}\int dx-\frac{1}{2}\int\cos 6x\,dx=$

$=\frac{1}{2}x-\frac{1}{2}\cdot\frac{1}{6}\sin 6x+C=\frac{x}{2}-\frac{\sin 6x}{12}+C.$ ■

3. $\int R(\cos x, \sin x)dx$, where $R(\cdot,\cdot)$ is a rational function of $\cos x$ and $\sin x$. For instance, $R(\cos x, \sin x) = \dfrac{1}{3+\sin x+\cos x}$.

The substitution $t = \tan\dfrac{x}{2}$ is offered to apply to taking the integral $\int R(\cos x, \sin x)dx$. The substitution $t = \tan\dfrac{x}{2}$ is called the ***universal trigonometric substitution***.

Express $\sin x$ and $\cos x$ in terms of t.

Representing $\sin x$ as a fraction $\dfrac{\sin x}{1}$ and using the double angle formula for $\sin x$: $\sin x = 2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$ and the trigonometric identity $\sin^2\dfrac{x}{2}+\cos^2\dfrac{x}{2}=1$ we have

$$\sin x = \frac{\sin x}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}.$$

Dividing by $\cos^2\dfrac{x}{2}$ results in

$$\text{sinx} = \frac{\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right):\cos^2\frac{x}{2}}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right):\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1}.$$

Since $t = \tan\dfrac{x}{2}$, we get

$$\sin x = \frac{2t}{t^2+1}.$$

In a similar way, we can get $\cos x = \dfrac{1-t^2}{1+t^2}$. Notice, that $dt = \dfrac{2}{\cos^2\frac{x}{2}}dx$. So according to the substitution $t = \tan\dfrac{x}{2}$ and the identity $\tan^2\dfrac{x}{2}+1 = \dfrac{1}{\cos^2\frac{x}{2}}$ it can be derived that $dx = \dfrac{2dt}{t^2+1}$.

Then

$$\int R(\sin x,\cos x)dx=\int R(\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2})\cdot\frac{2}{1+t^2}dt.$$

As a result we receive the integral of a rational fraction in terms of t. That's why this substitution is known as a ***rationalizing substitution***.

Example 4.40. Find $\int\frac{dx}{3+\sin x+\cos x}$.

□ $$\int\frac{dx}{3+\sin x+\cos x}=\left|t=\tan\frac{x}{2}\right|=\int\frac{\frac{2dt}{1+t^2}}{3+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{dt}{t^2+t+2}=\int\frac{dt}{\left(t^2+2\cdot\frac{1}{2}\cdot t+\frac{1}{4}\right)-\frac{1}{4}+2}=$$

$$=\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{7}{4}}=\left|d\left(t+\frac{1}{2}\right)=\left(t+\frac{1}{2}\right)'dt=dt\right|=\int\frac{d\left(t+\frac{1}{2}\right)}{\left(t+\frac{1}{2}\right)^2+\frac{7}{4}}=\left|u=t+\left(t+\frac{1}{2}\right)\right|=$$

$$\int\frac{du}{u^2+\frac{7}{4}}=\frac{2}{\sqrt{7}}\arctan\frac{2u}{\sqrt{7}}+C=\frac{2}{\sqrt{7}}\arctan\frac{2\left(t+\frac{1}{2}\right)}{\sqrt{7}}+C=\frac{2}{\sqrt{7}}\arctan\frac{2\left(\tan\frac{x}{2}+\frac{1}{2}\right)}{\sqrt{7}}+C.$$ ■

Example 4.41. Find $\int\frac{1}{5+\cos x}dx$.

□ $$\int\frac{1}{5+\cos x}dx=\left|\begin{array}{l}t=\mathrm{tg}\frac{x}{2}\\ \cos x=\frac{1-t^2}{1+t^2}\\ dx=\frac{2}{1+t^2}dt\end{array}\right|=\int\frac{1}{5+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}dt=\int\frac{2\cdot dt}{5+5t^2+1-t^2}=\int\frac{2}{4t^2+6}dt=$$

$$=\frac{2}{4}\int\frac{1}{t^2+\frac{3}{2}}dt=\frac{1}{2}\cdot\sqrt{\frac{2}{3}}\arctan\frac{t}{\sqrt{3/2}}+C=\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\tan\frac{x}{2}\right)+C.$$ ■

Remark 4.10.

If $R(-\sin x,-\cos x)=R(\sin x,\cos x)$, it is advisable to apply another substitution: $t=\tan x$ or $t=\cot x$.

Example 4.42. Find $\int\frac{dx}{1+\sin^2 x}$.

□ Since $R(-\sin x,-\cos x)=\frac{1}{1+(-\sin x)^2}=R(\sin x,\cos x)$ we should apply the substitution $t=\tan x$. Thus

$$\int \frac{dx}{1+\sin^2 x} = \left| \begin{array}{l} t = \tan x, \\ \text{dt} = \frac{dt}{1+t^2} \\ \sin x = \frac{t}{\sqrt{1+t^2}} \end{array} \right| = \int \frac{\frac{dt}{1+t^2}}{1+\frac{t^2}{1+t^2}} = \int \frac{dt}{1+2t^2} = \int \frac{dt}{1+\left(\sqrt{2}t\right)^2} = \left| \begin{array}{l} d\left(\sqrt{2}t\right) = \left(\sqrt{2}t\right)' dt = \\ = \sqrt{2}dt \end{array} \right| =$$

$$= \int \frac{\frac{1}{\sqrt{2}}\left(\sqrt{2}dt\right)}{1+\left(\sqrt{2}t\right)^2} = \frac{1}{\sqrt{2}} \int \frac{d\left(\sqrt{2}t\right)}{1+\left(\sqrt{2}t\right)^2} = \frac{1}{\sqrt{2}} \arctan\sqrt{2}t + C = \frac{1}{\sqrt{2}} \arctan\left(\sqrt{2}\tan x\right) + C. \ \blacksquare$$

Example 4.43. Find $\int \frac{1}{\sin^2 x \cdot \cos^4 x} dx$.

□ $R(\sin x, \cos x) = \frac{1}{\sin^2 x \cdot \cos^4 x}$ satisfies the condition:

$$R(-\sin x, -\cos x) = R(\sin x, \cos x).$$

We use the substitution $t = \tan x$ and the following trigonometric identities:

$$\frac{1}{\sin^2 x} = \cot^2 x + 1 = \frac{1}{\tan^2 x} + 1 \ ,$$

$$\frac{1}{\cos^2 x} = \tan^2 x + 1 .$$

Then

$$\int \frac{1}{\sin^2 x \cdot \cos^4 x} dx = \int \frac{1}{\sin^2 x} \cdot \frac{1}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx = \left| \begin{array}{l} t = \text{tg} x \\ dt = \frac{dx}{\cos^2 x} \end{array} \right| =$$

$$= \int \left(\frac{1}{t^2} + 1\right)\left(t^2 + 1\right) dt = \int \left(1 + \frac{1}{t^2} + t^2 + 1\right) dt = \int \left(2 + t^{-2} + t^2\right) dt = 2t + \frac{t^{-1}}{-1} + \frac{t^3}{3} + C = 2\tan x -$$

$$- \frac{1}{\tan x} + \frac{\tan^3 x}{3} + C = 2\tan x - \cot x + \frac{\tan^3 x}{3} + C . \ \blacksquare$$

Integration of irrational expressions

If some of terms involved in the numerator or in the denominator of a rational function replace with roots of rational fractions including polynomials then the obtained function is called an ***irrational function***. For example, the function $f(x) = \frac{1}{\sqrt{x} + \sqrt[3]{x}}$ is an irrational function.

In some cases integrals of irrational functions can be converted into integrals of rational functions or in other words integrands can rationalized by use of a substitution. The following cases can be distinguished:

1. $\int R\left(\sqrt[n]{x^m},\sqrt[q]{x^p},...,\sqrt[s]{x^l}\right)dx$.

The integrand $R\left(\sqrt[n]{x^m},\sqrt[q]{x^p},...,\sqrt[s]{x^l}\right)$ can be transformed into a rational function by means of the substitution $x=t^k$, where k is the least common multiply of the indexes $n,q,..,s$ ($LCM(n,q,..,s)$).

Example 4.44. Find $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

□ The integrand $\frac{1}{\sqrt{x}+\sqrt[3]{x}}$ involves one square root and one cubic root. So $n=2,q=3$. Then we should use the substitution $x=t^6$ as $6=LCM(2,3)$.

$$\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}=\begin{vmatrix}x=t^6,\\ dx=6t^5dt\end{vmatrix}=\int\frac{6t^5dt}{t^3+t^2}=\int\frac{6t^5dt}{t^2(t+1)}=6\int\frac{t^3dt}{t+1}=6\left[\int\frac{(t^3+1)-1dt}{t+1}\right]=$$

$$=6\int\frac{(t+1)(t^2-t+1)-1}{t+1}dt=6\left[\int(t^2-t+1)dt-\int\frac{dt}{t+1}\right]=6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right]+C=$$

$$=2t^3-3t^2+6t-6\ln|t+1|+C=2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\ln\left|\sqrt[6]{x}+1\right|+C.\ ■$$

2. $\int R\left(\sqrt[n]{(ax+b)^m},\sqrt[q]{(ax+b)^p},...,\sqrt[s]{(ax+b)^l}\right)dx,\ a,b=\text{const}$.

Using the substitution $ax+b=t^k,\ k=LCM(n,q,..,s)$ leads the integrand $R\left(\sqrt[n]{(ax+b)^m},\sqrt[q]{(ax+b)^p},...,\sqrt[s]{(ax+b)^l}\right)$ to a rational function.

Example 4.45. Find $\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}$.

□ Let $2x-1=t^4\Rightarrow x=\frac{1}{2}(t^4+1)$. Then $\sqrt{2x-1}=t^2$ and $\sqrt[4]{2x-1}=t$, $dx=\frac{1}{2}\cdot 4t^3dt$.

Let's eliminate x from the given integral and take the obtained one:

$$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\int\frac{2t^3dt}{t^2-t}=\int\frac{2t^2dt}{t-1}.$$

The last integral contains the improper fraction. We transform the integrand in such way:

$$\frac{2t^2}{t-1}=2\frac{(t^2-1)+1}{t-1}=2\left(\frac{t^2-1}{t-1}+\frac{1}{t-1}\right)=2\left(\frac{(t-1)(t+1)}{t-1}+\frac{1}{t-1}\right)=2\left(t+1+\frac{1}{t-1}\right).$$

Return to integrating:

$$\int\frac{2t^2dt}{t-1}=\int 2\left(t+1+\frac{1}{t-1}\right)dt=2\left(\frac{t^2}{2}+t+\ln|t-1|\right)+C.$$

Making the reverse substitution $t=\sqrt[4]{2x-1}$:

$$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\sqrt{2x-1}+2\sqrt[4]{2x-1}+2\ln|\sqrt[4]{2x-1}-1|+C.\blacksquare$$

Exercises

Find the integrals given below:

1. $\int\frac{\cos^3 x}{\sin^2 x}dx$;

2. $\int\sin^4\frac{x}{2}$;

3. $\int\sin\frac{x}{2}\sin\frac{x}{5}dx$;

4. $\int\cos^3 xdx$;

5. $\int\mathrm{tg}^4 3xdx$;

6. $\int\frac{dx}{\sqrt{\left(4-x^2\right)^3}}$;

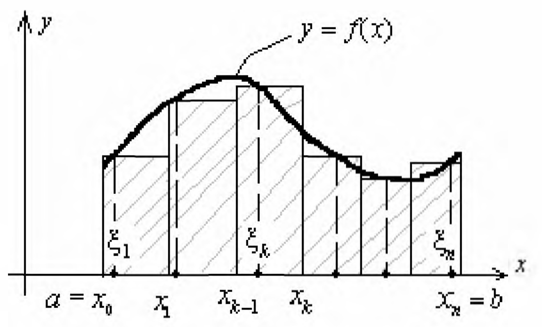
7. $\int\frac{\sqrt{x^2+16}}{x}dx$;

8. $\int\sin 2x\cos 3xdx$.

4.2. DEFINITE INTEGRALS AND THEIR APPLICATIONS

Definition and geometrical interpretation

Consider the problem of calculating the area of a region S in the coordinate plane, bounded by vertical lines with x-intercepts a and b, the x-axis and the graph of a function f, which is continuous and nonnegative on a closed interval $[a,b]$ (pic. 4.1).



Pic. 4.1

For convenience we shall refer to S as ***the region under the graph of*** f ***from*** a ***to*** b. Our goal is to define the area of S. In other words, we wish to calculate the ***"area under a curve"***.

Suppose, the area exists. Let S_a^b be the value of the area. Note, that $S_a^b=(b-a)\cdot f_0$ if $f(x)=f_0=\text{const}$ on $[a, b]$. It's obvious that this formula is not valid to evaluate the area under the curve when f is an arbitrary function. Let's begin by dividing the interval $[a, b]$ into n subintervals so that $[a,b]=[x_0,x_1]\cup[x_1,x_2]\cup\ldots\cup[x_{n-1},x_n]$. This can be accomplished by choosing numbers $x_0, x_1, x_2, \ldots, x_n$, where $a=x_0$, $b=x_n$ and $x_{k-1}<x_k$ for any $k=1,\ldots,n$. The set of these points is called a ***partition*** of the interval $[a, b]$ into a finite number of subintervals. The length of each subinterval $[x_{k-1}, x_k]$ is denoted by Δx_k and $\Delta x_k = x_k - x_{k-1}$. Note that $x_k = x_{k-1}+\Delta x_k$. On each subinterval we choose an arbitrary point $\xi_k\in[x_{k-1},x_k]$, $k=1,..,n$. The product $f(\xi_k)\cdot\Delta x_k$ is equal to the area of the rectangle of width Δx_k and height $f(\xi_k)$. Then, the wished area is approximately equal to the following sum:

$$S_a^b \approx \sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k . \tag{4.6}$$

The expression $\sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k$ is called the ***Riemann sum*** or ***the integral sum***.

Let $d = \max_k \Delta x_k$, where d is called ***the diameter*** of the partition. It is clear that with decreasing d, the accuracy of calculating the area increases. More rectangles of smaller width lead to a better approximation. So we determine the area as a limit of integral sums as $n \to \infty$ provided that $d \to 0$

$$S_a^b = \lim_{\substack{n\to\infty \\ (d\to 0)}} \sum_{k=1}^{n} f(\xi_k)\cdot \Delta x_k . \qquad (4.7)$$

Example 4.46. Find the area under the curve $y = x^2,\ x \in [0, 2]$.

□ Firstly we should form the integral sum $\sum_{k=1}^{n} f(\xi_k)\cdot \Delta x_k$. Let $\Delta x_k = \frac{b-a}{n} = \frac{2}{n}$, $\xi_k = x_k = \frac{2}{n}\cdot k.$ Then, $d = \frac{2}{n}$, $f(\xi_k) = (\xi_k)^2 = \left(\frac{2k}{n}\right)^2 = \frac{4k^2}{n^2}$, $\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k=1}^{n} \frac{4k^2}{n^2}\cdot\frac{2}{n} = \frac{8}{n^3}\cdot\sum_{k=1}^{n} k^2.$ Using the formula $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^3 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, we get

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} = \frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} = \frac{4}{3}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

$$S_0^2 = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{n\to\infty} \frac{4}{3}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{8}{3}. \blacksquare$$

The case of an arbitrary continuous function. Now we digress from the specific task and consider some function $f(x)$, which is continuous on $[a, b]$. Let $\sum_{k=1}^{\infty} f(\xi_k)\Delta x_k$ be the corresponding integral sum.

Def.: If a limit of Riemann sums exists and doesn't depend on a partition of $[a, b]$ and choice of points ξ_k, it is called the ***definite integral*** of $f(x)$ over $[a, b]$ and denoted by:

$$\int_a^b f(x)dx = I = \lim_{\substack{n\to\infty \\ (d\to 0)}} \sum_{k=1}^{n} f(\xi_k)\Delta x_k,$$

where

$\int_a^b f(x)dx$ is the definite integral of f from a do b,

I – the value of the definite integral,

$f(x)dx$ – the integrand,

$f(x)$ – the integrand function or the integrand,
a – the lower limit of integration,
b – the upper limit of integration.

Def.: The function f is called ***integrable on*** $[a, b]$, if the definite integral $\int_a^b f(x)dx$ exists.

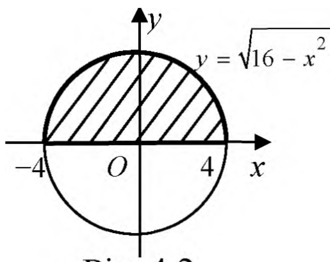
Remark 4.11

In this section we only consider integrals of bounded functions over bounded segments – so-called ***proper integrals***. Integrals like $\int_a^{\infty} e^{-x}dx$ and $\int_a^b \frac{dx}{x-a}$ are examples of improper integrals. They will be observed later.

Geometric interpretation

As it was shown above, the definite integral of a nonnegative function f, is equal to the area under the graph of f:

$$\int_a^b f(x)dx = S_a^b .$$



Pic. 4.2

Example 4.47. Evaluate $\int_{-4}^{4} \sqrt{16 - x^2}dx$.

□ Notice, the wished region whose area should be found is upper semicircle with center at $(0,0)$ and radius 4 (pic. 4.2). Indeed, the integrand $y = \sqrt{16 - x^2}$ or $x^2 + y^2 = 16$. Thus,

$$\int_{-4}^{4} \sqrt{16 - x^2}dx = \frac{1}{2}S_{\text{circle}} = \frac{1}{2}\frac{\pi \cdot 4^2}{2} = 4\pi \ .■$$

Connection between integrability, continuity and monotonocity

Proposition 4.3. If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ is integrable on $[a,b]$.

Continuity $\Longrightarrow$ Integrability

Proposition 4.4. If $f(x)$ is monotonous and bounded on a closed interval $[a,b]$, then $f(x)$ is integrable on $[a,b]$.

Monotonocity + Boundedness $\Longrightarrow$ Integrability

Proposition 4.5. If $f(x)$ is integrable on [*a*,*b*], then $f(x)$ is bounded on [*a*,*b*].

Integrability $\Longrightarrow$ Boundedness

Remark 4.12

If $f(x)$ isn't bounded on a closed internal [*a*, *b*], then $f(x)$ isn't integrable on [*a*, *b*] (in the sense of existence of a definite integral (see the definition given above))

Proposition 4.3 and 4.4 are sufficient conditions of integrability $f(x)$ on [*a*, *b*]. Proposition 4.5 can be considered as the necessary condition of integrability $f(x)$ on [*a*, *b*].

Properties of definite integrals

Let functions $f(x)$ and $g(x)$ be integrable on concerned closed intervals. Properties listed below can be directly derived from the definition.

1. If $f(a)$ exists, then $\int_a^a f(x)dx=0$.

2. $\int_a^b f(x)dx=-\int_b^a f(x)dx$.

3. The linear property: $\int_a^b (\alpha f(x)+\beta g(x))dx=\alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx$, where $\alpha, \beta \in \mathbb{R}, \alpha \neq 0, \beta \neq 0$.

4. The additive property: for any *a*, *b*, *c* $\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$.

5. If $f(x)=A$, $A=$ const, then $\int_a^b f(x)dx=A(b-a)$.

Remark 4.13

According to property 5, $\int_{-5}^{6} dx=\left|\begin{matrix} f(x)=1 \Rightarrow A=1 \\ a=-5,\ b=6 \end{matrix}\right|=6-(-5)=11$.

6. If $f(x)\geq 0$ on [*a*, *b*], then $\int_a^b f(x)dx\geq 0$.

7. If $f(x)\geq g(x)$ on [*a*, *b*], then $\int_a^b f(x)dx\geq\int_a^b g(x)dx$.

8. If $f(x)$ is integrable on [*a*, *b*], then $|f(x)|$ is also integrable on [*a*, *b*] and

$$\int_a^b |f(x)|dx \ge \left|\int_a^b f(x)dx\right|.$$

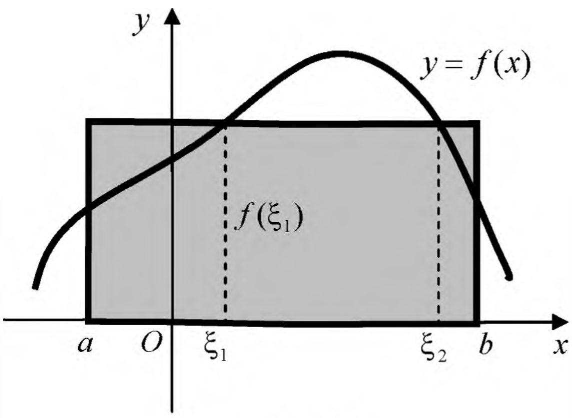
9. **The Mean Value Theorem for definite integrals (MVT).** *If* $f(x)$ *is continuous on* $[a, b]$, *then there is a number* $\xi \in [a,b]$, *such that*

$$\int_a^b f(x)dx = f(\xi)(b-a).$$

Remark 4.14

The number ξ is not necessarily unique (see pic. 4.3).

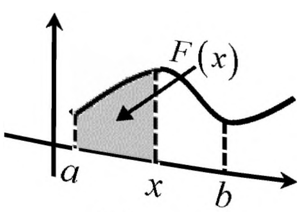
The MVT has an interesting ***geometric interpretation***, if $f(x) \ge 0$ on $[a,b]$. $\int_a^b f(x)dx$ is the area under the graph of f from a to b.



Pic. 4.3

The right-hand product $f(\xi)(b-a)$ is the area of a rectangular region bounded by a horizontal line $y = f(\xi)$, the x-axis, and lines $x = a$ and $x = b$. According to the MVT, the areas of these figures are equal (pic. 4.3).

Fundamental theorem of calculus



Pic. 4.4

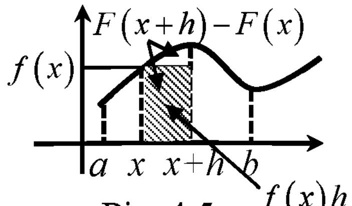
Evaluating definite integrals by means of taking a limit of the Reimann sum is very complicated for the most part. So principally another way of calculating is shown below.

Suppose f is integrable on a closed interval $[a,b]$, then $\int_a^x f(t)\,dt$ defines a new function F of x:

$$F(x) = \int_a^x f(t)\,dt,\ x \in [a,b] \text{ (pic. 4.4)}$$

It can be proved that if f is continuous, F is differentiable and moreover

$$\frac{d}{dx}F(x)=f(x).$$



Pic. 4.5

Indeed, $F'(x)$ can be approximated by $\dfrac{F(x+h)-F(x)}{h}$, where h is enough small and $x+h\in[a,b]$. $F(x+h)-F(x)$ is approximately equal to the area the rectangle with width h and height $f(x)$ (see pic. 4.5). Thus,

$$\frac{d}{dx}F(x)=F'(x)=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}=\lim_{h\to 0}\frac{hf(x)}{h}=f(x).$$

Fundamental Theorem of Calculus, Part I.

If f is continuous on $[a, b]$, then $F(x)=\int\limits_a^x f(t)\,dt,\ x\in[a,b]$ is continuous on $[a, b]$ and differentiable on (a,b) and

$$F'(x)=\frac{d}{dx}\left(\int\limits_a^x f(t)\,dt\right)=f(x).$$

Fundamental Theorem of Calculus, Part II.

If f is continuous on $[a, b]$ and $F(x)$ is any antiderivative of f on $[a, b]$, then

$$\int\limits_a^b f(x)dx=F(x)\Big|_a^b=F(b)-F(a).$$

This formula is called ***the Newton–Leibniz formula***.

We indicate ***three ways of calculating*** the definite integrals:
– by the limit of Riemann sums,
– by the geometric sense of the integral,
– by the Newton–Leibniz formula.

For example 4.46 the area under the graph of x^2 from 0 to 2 found above as the limit of the Reimann sum can be also calculated as follows:

$$S_0^2=\int\limits_0^2 x^2dx=\frac{x^3}{3}\Bigg|_0^2=\frac{2^3}{3}-\frac{0^3}{3}=\frac{8}{3}.$$

Hence, x^2 is an integrable function on [0; 2].

Example 4.48. Find $\int\limits_1^4\left(\frac{1}{x^2}+\sqrt{x}\right)dx.$

□ According to property 3 and the Newton–Leibniz formula,

$$\int_1^4\left(\frac{1}{x^2}+\sqrt{x}\right)dx=\int_1^4\left(x^{-2}+x^{1/2}\right)dx=\int_1^4 x^{-2}dx+\int_1^4 x^{1/2}dx=\left.\frac{x^{-1}}{-1}\right|_1^4+\left.\frac{x^{3/2}}{3/2}\right|_1^4=$$

$$=-\left.\frac{1}{x}\right|_1^4+\left.\frac{2}{3}x^{3/2}\right|_1^4=-\frac{1}{4}-(-1)+\frac{2}{3}\left(4^{3/2}-1\right)=-\frac{1}{4}+1+\frac{2}{3}(8-1)=$$

$$=\frac{3}{4}+\frac{14}{3}=\frac{9+56}{12}=\frac{65}{12}=5\frac{5}{12}.\ ■$$

Example 4.49. Find $\int_0^{\pi}\sqrt{\frac{1+\cos(2x)}{2}}\,dx$.

□ Since $\frac{1+\cos(2x)}{2}=\cos^2 x$, we get

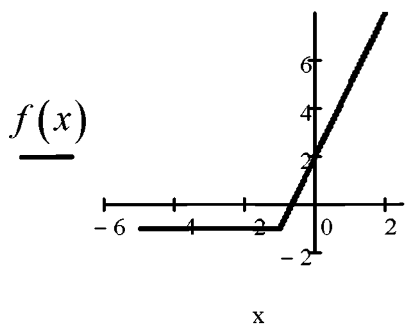
$$\int_0^{\pi}\sqrt{\frac{1+\cos(2x)}{2}}\,dx=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}|\cos x|dx.$$

Taking into consideration the fact that $|\cos x|=\begin{cases}\cos x, \text{ if } x\in\left[0,\frac{\pi}{2}\right],\\ -\cos x, \text{ if } x\in\left[\frac{\pi}{2},\pi\right]\end{cases}$,

$$\int_0^{\pi}|\cos x|dx=\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}(-\cos x)dx=\sin x\Big|_0^{\pi/2}-\sin x\Big|_{\pi/2}^{\pi}=1+1=2.■$$

Example 4.50. Find $\int_{-5}^{2}f(x)dx$, if $f(x)=\begin{cases}-1, & \text{if } x\in[-5,-1],\\ 2+3x, & \text{if } x\in(-1,2].\end{cases}$

□ The given function f is a piecewise-defined function, whose graph is shown in pic. 4.6.



Pic. 4.6

Function $f(x)$ is continuous on $[-5,2]$. Indeed, -1 and $2+3x$ are continuous on $[-5,-1)$ and $(-1,2]$ respectively as basic elementary functions. Further, let's examine $f(x)$ for continuity at $x=-1$. The left-hand limit $\lim\limits_{x\to -1-0} f(x)=-1$, the right-hand limit $\lim\limits_{x\to -1+0} f(x)=-1$ and moreover $f(-1)=-1$. Thus, being based on the approach applied in the previous example,

$$\int_{-5}^{2} f(x)dx=\int_{-5}^{-1}(-1)dx+\int_{-1}^{2}(2+3x)dx=(-1)\cdot(-1-(-5))+\left(2x+\frac{3x^2}{2}\right)\Bigg|_{-1}^{2}=$$

$$=-4+\left(2\cdot 2+\frac{3\cdot 2^2}{2}\right)-\left(2\cdot(-1)+\frac{3(-1)^2}{2}\right)=-4+10+\frac{1}{2}=\frac{13}{2}.\blacksquare$$

Example 4.51. Find $\int_{-\pi}^{2\pi} f(x)dx$, if $f(x)=\begin{cases}\cos x, \text{ if } x\in[-\pi;\, 0],\\ \sin x, \text{ if } x\in(0;\ 2\pi].\end{cases}$

□ The given function $f(x)$ is a piecewise-defined function. It is continuous at every point in segment $[-\pi;\, 2\pi]$ except $x=0$, because $f(0)=1\neq f(0+0)=$ $=\lim\limits_{x\to 0+0}\sin x=0$. Function $f(x)$ has a jump discontinuity at $x=0$.

To take the integral we should represent it as a sum of two integrals over $[-\pi;\, 0]$ and $(0;\ 2\pi]$:

$$\int_{-\pi}^{2\pi} f(x)dx=\int_{-\pi}^{0}\cos x\,dx+\int_{0}^{2\pi}\sin x\,dx=\sin x\Big|_{-\pi}^{0}+(-\cos x)\Big|_{0}^{2\pi}=$$

$$=(\sin 0-\sin(-\pi))-(\cos 2\pi-\cos 0)=(0-0)-(1-1)=0.\blacksquare$$

Integration by substitution

Proposition 4.6. *Suppose $t=\psi(x)$ has continuous derivative $\psi'(x)$ on $[\alpha,\beta]$, $f(t)$ is a continuous function on $[a,b]$, where $\psi(\alpha)=a$, $\psi(\beta)=b$. Then*

$$\int_{\alpha}^{\beta} f(\psi(x))\cdot\psi'(x)dx=\int_{a}^{b} f(t)dt. \tag{4.8}$$

This method is called ***integration by substitution*** for the definite integral.

Example 4.52. Find $\int_{0}^{1} x\cdot(2-x^2)^5 dx.$

□ Since $d(2-x^2)=-2xdx$, let $t=\psi(x)=2-x^2$. Changing variable of integration implies changing limits of integration. Thus, the new lower limit of integration $a=\psi(0)=2-0^2=2$, the new upper limit of integration $b=\psi(1)=2-1^2=1$. Then

$$\int_0^1 x(2-x^2)^5 dx = \left| \begin{array}{l} t=2-x^2 \\ dt=-2xdx \\ xdx=-\frac{1}{2}dt \\ \alpha=0 \Rightarrow a=2 \\ \beta=1 \Rightarrow b=1 \end{array} \right| = \int_2^1 t^5\left(-\frac{1}{2}dt\right) = -\frac{1}{2}\int_2^1 t^5 dt = \frac{1}{2}\int_1^2 t^5 dt = \frac{1}{2}\cdot\frac{t^6}{6}\bigg|_1^2 =$$

$$= \frac{1}{12}\left(2^6-1^6\right) = \frac{64-1}{12} = \frac{63}{12} = \frac{21}{4}.$$

Note, that we have to apply property 2 because the new upper limit of integration is less the new lower limit: $a=2>b=1$.■

Example 4.53. Find $\int_0^{\ln 3} \frac{e^{3x}}{1+e^{3x}}dx$.

□ To take the integral we should make the substitution $t=1+e^{3x}$ because the expression for $dt=3e^{3x}dx$ is involved in the integrand up to a constant.

$$\int_0^{\ln 3} \frac{e^{3x}}{1+e^{3x}}dx = \left| \begin{array}{l} t=1+e^{3x} \\ dt=3e^{3x}dx \\ e^{3x}dx=\frac{dt}{3} \\ \alpha=0 \Rightarrow a=2 \\ \beta=\ln 3 \Rightarrow b=28 \end{array} \right| = \int_2^{28}\frac{dt}{3t} = \frac{1}{3}\ln|t|\,\bigg\|_2^{28} = \frac{1}{3}(\ln 28-\ln 2) = \frac{1}{3}\ln 14. \ ■$$

Example 4.54. Find $\int_0^{c^2} \frac{dx}{2\sqrt{x}(x+c^2)}$.

□ The integrand contains $\frac{dx}{2\sqrt{x}}$ that is $d\sqrt{x}$, so we can take $\sqrt{x}$ for the new variable of integration t. Then

$$\int_0^{c^2} \frac{dx}{2\sqrt{x}(x+c^2)} = \left| \begin{array}{l} t=\sqrt{x} \\ dt=\frac{dx}{2\sqrt{x}} \\ \alpha=0 \Rightarrow a=0 \\ \beta=c^2 \Rightarrow b=c \end{array} \right| = \int_0^c \frac{dt}{t^2+c^2} = \frac{1}{c}\arctan\frac{t}{c}\bigg|_0^c = \frac{1}{c}\left(\text{arctang}\frac{c}{c}-\arctan 0\right) = \frac{\pi}{4c}. \ ■$$

Integration by parts

Proposition 4.7. *Let* $u(x)$ *and* $v(x)$ *be functions with continuous derivatives on a closed interval* $[a,b]$. *Then*

$$\int_a^b u\cdot dv = u\cdot v\big|_a^b - \int_a^b v\cdot du. \qquad (4.9)$$

This method is called ***integration by parts*** for the definite integral.

Example 4.55. Find $\int_0^{\frac{\pi}{2}} x\cdot\cos x dx.$

□ According to case 8 with $u=$x, $dv=\cos x\,dx$ (see subsection "integration by parts" for the indefinite integral), we have

$$\int_0^{\frac{\pi}{2}} x\cdot\cos x dx = \left| \begin{matrix} u = x \Rightarrow du = dx \\ dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x \end{matrix} \right| = x\cdot\sin x\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx =$$

$$= \frac{\pi}{2}\cdot\sin\frac{\pi}{2} - 0 + \cos x\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1.\ ■$$

Example 4.56. Find $\int_0^1 (4e^x - 5)x dx$.

□ The integrand $(4e^x-5)x$ corresponds to case 7 with $u=$x, $dv=\left(4e^x-5\right)dx$ (see subsection "integration by parts" for the indefinite integral), so applying (4.9)

$$\int_0^1 (4e^x-5)xdx = x(4e^x-5x)\Big|_0^1 - \int_0^1 (4e^x-5x)dx = (4e-5) - 0 - \left(4e^x - 5\cdot\frac{x^2}{2}\right)\Bigg|_0^1 =$$

$$= 4e - 5 - (4e - 2{,}5) + (4 - 0) = 1{,}5.\ ■$$

Example 4.57. Find $\int_0^{\pi} x^2 \sin x dx$.

□ Unlike the previous example, the formula (4.9) should be applied consecutively twice.

$$\int_0^{\pi} x^2\sin x dx = \left| \begin{matrix} u = x^2 \Rightarrow du = 2xdx \\ dv = \sin x dx \Rightarrow v = \int \sin x dx = -\cos x \end{matrix} \right| = x^2(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} 2x(-\cos x)dx$$

$$= \pi^2 + 0 + 2\int_0^{\pi} x\cos x dx = \left| \begin{matrix} u = x \Rightarrow du = dx \\ dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x \end{matrix} \right| = \pi^2 + 2(x\sin x\Big|_0^{\pi} -$$

$$-\int_0^{\pi} \sin x dx) = \pi^2 + 2(0 - 0 + \cos x\Big|_0^{\pi}) = \pi^2 + 2(-1-1) = \pi^2 - 4.■$$

Integration of even and odd functions over intervals with symmetry with respect to the origin

Suppose, f is an integrable function on $[-l,l]$.
If $f(-x) = -f(x)$, i.e. f is an odd function, then it is easy to prove that

$$\int_{-l}^{l} f(x)dx = 0,\ \ l \in R.$$

If $f(-x)=f(x)$, i.e. f is an even function, then

$$\int_{-l}^{l} f(x)dx = 2\cdot\int_{0}^{l} f(x)dx,\ l\in R.$$

Example 4.58. Find $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx$.

□ There are two ways of taking the integral:

1) Note, $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ is a symmetric interval, $\sin^3 x$ is an odd function: $\sin^3(-x)=-\sin^3(x)$. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx = 0.$$

2) Using the trigonometric functions integrating method, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3 x dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\cdot \sin x dx = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-\cos^2 x)\cdot d\cos x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2 x - 1)\cdot d\cos x =$$

$$= \left(\frac{(\cos x)^3}{3} - \cos x\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{\left(\cos\left(\frac{\pi}{2}\right)\right)^3}{3} - \cos\left(\frac{\pi}{2}\right)\right) - \left(\frac{\left(\cos\left(-\frac{\pi}{2}\right)\right)^3}{3} - \cos\left(-\frac{\pi}{2}\right)\right) =$$

$$= \left|\cos\left(-\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)\right| = 0. \blacksquare$$

Example 4.59. Find $\int_{-10}^{10} \cos nx dx,\ n\in Z, n\neq 0.$

□ The closed interval $[-10;10]$ is a symmetric interval, $\cos nx$ is an even function: $\cos(n(-x)) = \cos nx$. Then

$$\int_{-10}^{10} \cos nx dx = 2\cdot\int_{0}^{10} \cos nx dx = 2\cdot\frac{\sin nx}{n}\Bigg|_{0}^{10} = \frac{2\sin 10n}{n} - 0 = \frac{2\sin 10n}{n}. \blacksquare$$

Exercises

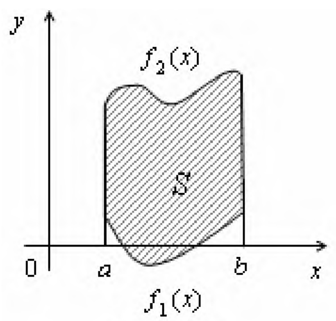
1. Find $\int_0^8 \left(\sqrt{2}+\sqrt[3]{x}\right)dx$.
2. Find $\int_1^4 \frac{1+\sqrt{x}}{x^2}dx$.
3. Find $\int_0^1 \frac{xdx}{x^2+3x+2}$.
4. Find $\int_4^5 x\sqrt{x^2-16}dx$.
5. Find $\int_0^{\sqrt{3}} \arctan x\,dx$.
6. Find $\int_{-\pi}^{\pi} x\sin x\cos x dx$.
7. Find $\int_0^1 \frac{dx}{x^2+4x+5}$.
8. Find $\int_1^3 \frac{dx}{x^3+x}$.
9. Find $\int_{-3}^3 x\sqrt{9-x^2}\,dx$.
10. Find $\int_{\pi}^{3\pi/2} \sin^4 x\cos^3 x\,dx$.

Applications to geometry

Using the definite integral we can find:
– areas of plane figures,
– arc lengths of plane and space curves,
– volumes of solids of revolution.

Areas of plane figures

1) ***Explicit boundary equations.*** Let $f_1(x), f_2(x)$ be continuous on $[a;b]$ and $f_1(x)\le f_2(x)$ (see pic. 4.7).



Pic. 4.7

Then the area between two curves is equal to S:

$$S=\int_a^b \left(f_2(x)-f_1(x)\right)dx.$$

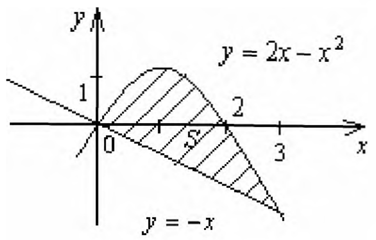
Example 4.60. Find the area of the region between the parabola $y = 2x - x^2$ and the straight line $y = -x$.

□ Graph these functions first and then find the intersection points (see pic. 4.8).

$$\begin{cases} y = 2x - x^2, \\ y = -x; \end{cases} \Rightarrow -x = 2x - x^2 \Rightarrow x^2 - 3x = 0 \Rightarrow \left[\begin{array}{l} x = 0, \\ x = 3. \end{array}\right.$$

The equation $y = -x$ describes the bisectrix of the second and the fourth coordinate quadrants while $y = 2x - x^2$ is an equation of opening up parabola whose vertex is located at the point with coordinates: $x_v = -\dfrac{2}{2\cdot(-1)} = 1$, $y_v = 2\cdot 1 - (1)^2$. In addition to this analyses we can find points where the parabola intersects the $x-$ axis ($x-$intercepts):

$$\begin{cases} y = 2x - x^2, \\ y = 0; \end{cases} \Rightarrow 2x - x^2 = 0 \Rightarrow \left[\begin{array}{l} x = 0, \\ x = 2. \end{array}\right.$$



Pic. 4.8

Applying the formula $S = \int\limits_a^b (f_2(x) - f_1(x))dx$, we get

$$S = \int\limits_0^3 (2x - x^2 - (-x))dx = \int\limits_0^3 (3x - x^2)dx = \left(\frac{3}{2}x^2 - \frac{x^3}{3}\right)\Bigg|_0^3 = \frac{27}{2} - 9 =$$

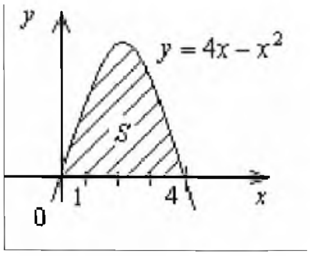
$$= \frac{27-18}{2} = \frac{9}{2} = 4{,}5. \blacksquare$$

Example 4.61. Find the area of the region between the parabola $y = 4x - x^2$ and the x-axis.

□ First of all draw the graph of the function $y = 4x - x^2$ and find points at which the parabola intersects the x-axis:

$$\begin{cases} y = 4x - x^2, \\ y = 0; \end{cases} \Rightarrow 4x - x^2 = 0 \Rightarrow \left[\begin{array}{l} x = 0, \\ x = 4. \end{array}\right.$$

The region whose area we wish to find is given in pic. 4.9.



Pic. 4.9

Then

$$S=\int_0^4\left(4x-x^2\right)dx=\left(2x^2-\frac{x^3}{3}\right)\Bigg|_0^4=32-\frac{64}{3}=\frac{32}{3}. \blacksquare$$

2) ***Boundary equations given parametrically.*** Let the region be bounded by a curve given by $x=x(t),\ \ y=y(t),\ \ t_0\le t\le t_1$, two lines $x=a, x=b$ and the x-axis. Then the area under the parametric curve is equal

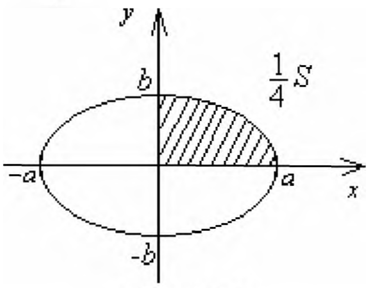
$$S=\int_{t_0}^{t_1} y(t)x'(t)dt,\ \ y(t)\ge 0.$$

where $x(t_0)=a,\ x(t_1)=b$.

Example 4.62. Find the area of the region bounded by the ellipse: $x=a\cos t$, $y=b\sin t$, $a>0,\ b>0$.

□ Taking into account symmetry of the region about coordinate axis we can evaluate a quarter of the area we want to find and then multiply the result by 4 (see pic. 4.10):

$$\frac{1}{4}S=\int_{t_a}^{t_b} b\sin t(a\cos t)'dt$$



Pic. 4.10

To define limits of integration we solve equations:

$$x=0\Leftrightarrow a\cos t=0\Rightarrow t_0=\frac{\pi}{2}.$$

$$x=a\Leftrightarrow a\cos t=a\Rightarrow t_1=0.$$

Substituting the limits of integration, we get

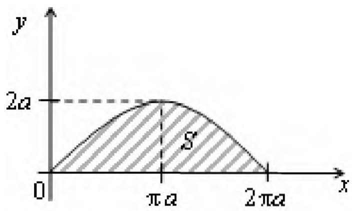
$$\frac{1}{4}S=\int_{\pi/2}^{0} b\sin t(a\cos t)'dt=-ab\int_0^{\pi/2}\sin t(-\sin t)dt=ab\int_0^{\pi/2}\sin^2 t\,dt=ab\int_0^{\pi/2}\frac{1-\cos 2t}{2}dt=$$

$$=\frac{ab}{2}(t-\frac{1}{2}\sin 2t)\Big|_0^{\pi/2}=\frac{ab}{2}\left(\frac{\pi}{2}-\frac{1}{2}\sin(2\cdot\frac{\pi}{2})\right)-0=\frac{\pi ab}{4}.$$

The area of entire region is equal to $S=\pi ab.$ ■

Example 4.63. Find the area of the region bounded by the x-axes and one arc of the cycloid $x=a(t-\sin t)$, $y=a(1-\cos t)$.

□ Let the interval of changing t be $[0;2\pi]$ that corresponds to one arc of the cycloid (see pic. 4.11).



Pic. 4.11

Then

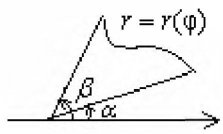
$$S=\int_0^{2\pi} a(1-\cos t)\big(a(t-\sin t)\big)'\,dt\int_0^{2\pi} a(1-\cos t)a(1-\cos t)dt=a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)dt=$$

$$=a^2\int_0^{2\pi}\left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt=a^2\int_0^{2\pi}\left(\frac{3}{2}-2\cos t+\frac{1}{2}\cos 2t\right)dt=$$

$$=a^2\left(\frac{3}{2}t-2\sin t+\frac{1}{4}\sin 2t\right)\Big|_0^{2\pi}=a^2\left(\frac{3}{2}2\pi-2\sin 2\pi+\frac{1}{4}\sin 4\pi-0\right)=3\pi a^2.$$ ■

3) ***Boundary equations in polar coordinates.*** In polar coordinates a curve is represented by an equation $r=r(\varphi)$, $\alpha\le\varphi\le\beta$.

Part of the plane enclosed between two rays $\varphi=\alpha$, $\varphi=\beta$ and an arc of the curve $r(\varphi)$, is called a ***curvilinear sector*** (see pic. 4.12).



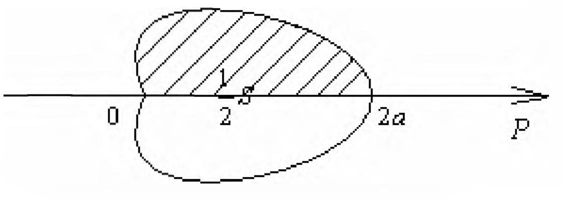
Pic. 4.12

The area S of the curvilinear sector is equal to

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\varphi)d\varphi.$$

Example 4.64. Find the area of the region enclosed inside the cardioid $r = a(1+\cos\varphi), a > 0$.

□ The cardioid has symmetry about the polar axis as $r(-\varphi) = a(1+\cos(-\varphi)) = a(1+\cos\varphi) = r(\varphi)$. The x-axes plays the role of the polar axis. So it's worthwhile to calculate one half of the desired area: $\frac{1}{2}S$, where $0 \le \varphi \le \pi$ (see pic. 4.13).



Pic. 4.13

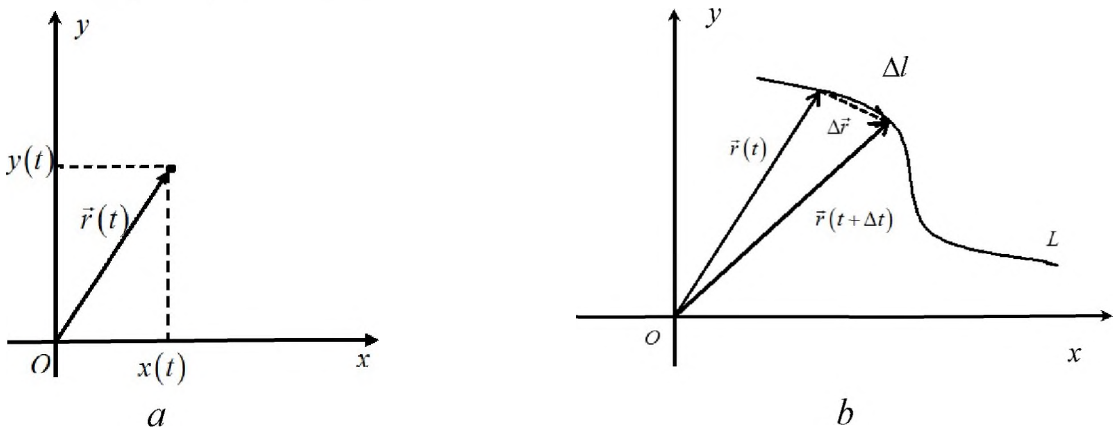
$$\frac{1}{2}S = \frac{1}{2}\int_0^{\pi} a^2(1+\cos\varphi)^2 d\varphi \Rightarrow S = a^2\int_0^{\pi}(1+2\cos\varphi+\cos^2\varphi)d\varphi =$$

$$= a^2\int_0^{\pi}\left(1+2\cos\varphi+\frac{1+\cos 2\varphi}{2}\right)d\varphi = a^2\int_0^{\pi}\left(\frac{3}{2}+2\cos\varphi+\frac{1}{2}\cos 2\varphi\right)d\varphi =$$

$$= a^2\left(\frac{3}{2}\varphi+2\sin\varphi+\frac{1}{4}\sin 2\varphi\right)\Big|_0^{\pi} = a^2\left(\frac{3}{2}\pi+2\sin\pi+\frac{1}{4}\sin 2\pi-0\right) = \frac{3\pi a^2}{2}. \ ■$$

Arc lengths of plane curves

1) Let a plane curve L be represented parametrically: $\begin{cases} x = x(t), \\ y = y(t), \end{cases}$ $t \in [t_0, t_1]$, $x(t)$ and $y(t)$ can be considered as coordinates of the radius vector of a point lying on the curve (pic. 4.14, *a*). It implies vector form of the curve representation: $\vec{r}(t) = (x(t), y(t))^T$.



Pic. 4.14

Denote arc length of the curve as l, then infinitesimal element of arc length is dl. Further let's find a derivative $\frac{dl}{dt}$. By the definition of a derivative we have $\frac{dl}{dt}=\lim_{\Delta t\to 0}\frac{\Delta l}{\Delta t}$. Moreover, Δl is approximately equal to $|\Delta\vec{r}|$ for enough small Δt (see pic.4.14, *b*). So $\lim_{\Delta t\to 0}\frac{\Delta l}{\Delta t}=\lim_{\Delta t\to 0}\frac{|\Delta\vec{r}|}{\Delta t}=\lim_{\Delta t\to 0}\frac{|\vec{r}(t+\Delta t)-\vec{r}(t)|}{\Delta t}$,where $\Delta\vec{r}=\vec{r}(t+\Delta t)-\vec{r}(t)=$ $=(x(t+\Delta t)-x(t),y(t+\Delta t)-y(t))^T=(\Delta x,\Delta y)^T$. As we know the length $|\vec{a}|$ of a vector $\vec{a}=(a_1,a_2)^T$ can be evaluated as $\sqrt{a_1^2+a_2^2}$.

Summarizing the above we get

$$\frac{dl}{dt}=\lim_{\Delta t\to 0}\frac{\sqrt{\Delta x^2+\Delta y^2}}{\Delta t}=\lim_{\Delta t\to 0}\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\Rightarrow$$

$$\Rightarrow dl=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt.$$

It's obvious that $l=\int dl$. More precisely we come to

$$l=\int_{t_0}^{t_1}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt. \qquad (4.10)$$

2) Let a plane curve L be represented by $y=f(x),\ a\le x\le b$. If we add the equation $x=x$ to the given equation and treat x as a parameter then applying the formula we have

$$l=\int_a^b\sqrt{\left(\frac{dx}{dx}\right)^2+\left(\frac{df(x)}{dx}\right)^2}dx=\int_a^b\sqrt{1+\left(\frac{df}{dx}\right)^2}dx. \qquad (4.11)$$

3) Let a plane curve L be given in polar coordinates: $r=r(\varphi),\ \alpha\le\varphi\le\beta$. Then taking $r(\varphi)\cos\varphi$ and $r(\varphi)\sin\varphi$ for x and y respectively and considering φ as a parameter lead us to

$$l=\int_\alpha^\beta\sqrt{r^2+\left(\frac{dr}{d\varphi}\right)^2}d\varphi. \qquad (4.12)$$

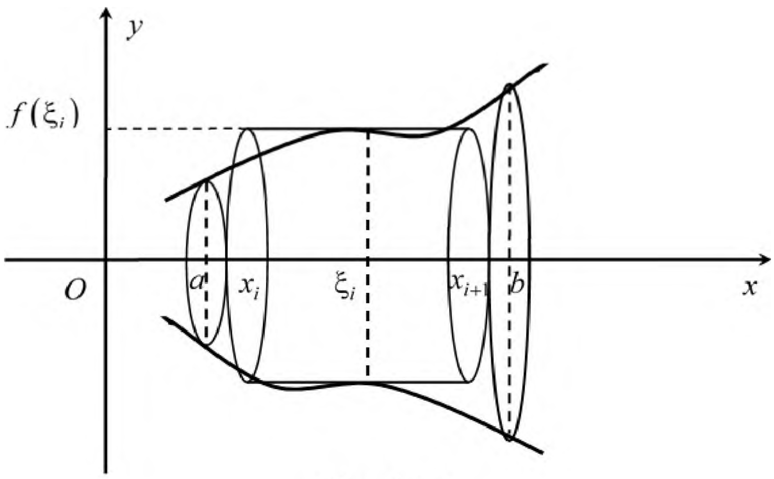
Example 4.65. Find arc length of the curve described by $y^2 = x^3$ between the origin $O(0,0)$ and $A\left(\frac{4}{3}, \frac{8\sqrt{3}}{9}\right)$.

□ The considered curve segment is located on the first coordinate quarter and represented by the equation $y = f(x) = x^{3/2}$. Since $f'(x) = \frac{3}{2}x^{1/2}$ and $0 \le x \le \frac{4}{3}$, applying the formula (4.11) we have

$$l = \int_0^{4/3} \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2} dx = \int_0^{4/3} \sqrt{1 + \frac{9}{4}x} dx = \frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg|_0^{4/3} = \frac{8}{27}\left(4^{3/2} - 1\right) = \frac{56}{27}. \blacksquare$$

Volumes of solids of revolution

Suppose, a continuous function $y = f(x)$ is defined on $[a,b]$ and keeps its sign on it. The problem is to find the volume V_x of a solid obtained by revolving a figure bounded by the graph of $f(x)$, $x = a, x = b, y = 0$ about the x -axis (see pic. 4.15).



Pic. 4.15

To get the answer the same approach as we applied when observing a concept of the definite integral is considered. Divide the interval $[a,b]$ into n subintervals: $[a,x_1],[x_1,x_2],..,[x_{n-1},b]$ and take a point ξ_i in each subinterval $[x_i,x_{i+1}]$, $i = 1,2,..,n$. Then form a cylinder whose height $\Delta x_i = x_{i+1} - x_i$ and base radius $f(\xi_i)$. Its volume is $\pi f^2(\xi_i)\Delta x_i$. So we can take $\sum_{i=1}^{n} \pi f^2(\xi_i)\Delta x_i$ for the approximation of the desired volume V_x. Note, that the smaller intervals we choose the better approximation we have. Thus,

$$V_x = \lim_{\substack{n\to\infty \\ (\max \Delta x_i \to 0)}} \sum_{i=1}^{n} \pi f^2(\xi_i)\Delta x_i = \pi\int_a^b f^2(x)dx. \quad (4.13)$$

Example 4.66. Find the volume of a solid obtained by revolving the plane region bounded by $y = e^x$, $y = 0$, $x = 0$, $x = 1$ about the x-axis.

□ By the formula (4.13)

$$V_x = \pi\int_0^1 \left(e^{-x}\right)^2 dx = \pi\left(-\frac{1}{2}e^{-2x}\right)\Bigg|_0^1 = \frac{\pi}{2}\left(1-\frac{1}{e^2}\right).■$$

Exercises

1. Find the area of the plane region bounded by
   a) $y = x^2, y = \frac{1}{x}, x = 3, y = 0$.
   b) $y = x^2, y = \frac{1}{x}, y = 4$. The region is located in the first quarter.
   c) $y = 4 - x^2, y = x^2 - 2x$.
   d) $y = \ln x, y = \ln(x+1), y = 1, y = -1$.
2. Find the area of region whose boundary is described by the parametric equations $\begin{cases} x = 2\sqrt{2}\cos t, \\ y = 5\sqrt{2}\sin t \end{cases}$ and $y = 5$ $(y \ge 5)$.
3. Find the area of region whose boundary is represented by the parametric equations $\begin{cases} x = 2 + 3\cos t, \\ y = 3 + 2\sin t. \end{cases}$
4. Find the area of the region with boundary given in polar coordinates $r = 5\cos\varphi$.
5. Find the area of the region with boundary given in polar coordinates $r = \sqrt{3}\sin\varphi$.
6. Find the area of the region with boundary given in polar coordinates $r = \cos\varphi$, $r = \sin\varphi$, $0 \le \varphi \le \frac{\pi}{2}$.
7. Find arc length of the curves described by
   a) $r = 2\varphi, 0 \le \varphi \le \frac{3}{4}$;
   b) $y = 4 - x^2, x = -2, x = 2$;
   c) $y = \ln x, x = \sqrt{3}, x = \sqrt{8}$;
   d) $x = t - \sin t, y = 1 - \cos t, 0 \le t \le 2\pi$.
8. Find the volume of solids obtained by revolving the plane regions bounded by:
   a) $y = x^2, y^2 = x$;
   b) $y = \cos 2x, y = 0, x = 0, x = \frac{\pi}{2}$.

4.3. IMPROPER INTEGRALS

Def.: Suppose, $f(x)$ defined on $[a,+\infty)$ is integrable on every closed interval $[a,\eta]$, contained in $[a,+\infty)$.

Then the quantity

$$\int_a^{+\infty} f(x)dx = \lim_{\eta\to+\infty}\int_a^{\eta} f(x)dx,$$

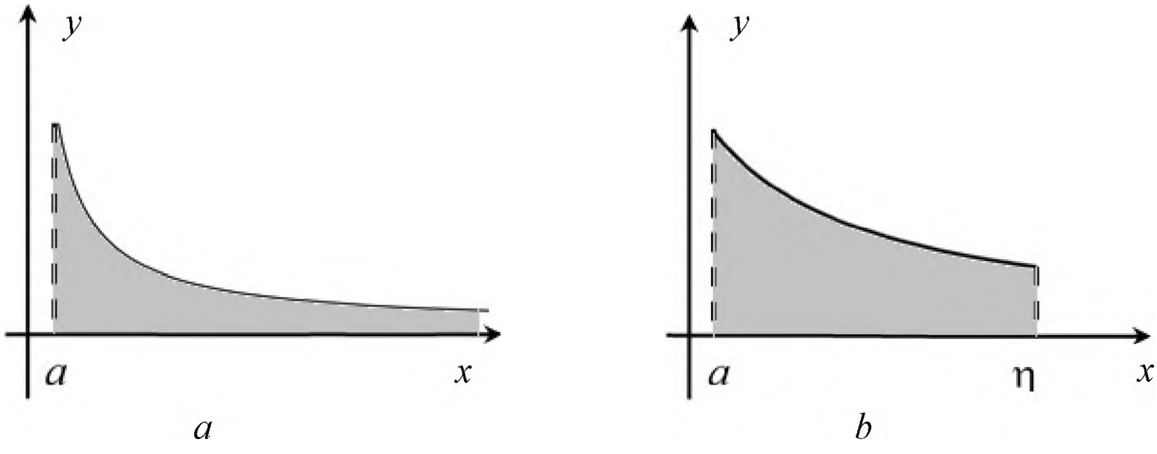
if this limit exists, is called an ***improper integral*** of the function $f(x)$ over the interval $[a,+\infty)$.

If this limit exists and equals a finite constant it is said that the improper integral ***converges*** and ***diverges*** otherwise.

Example 4.67. Consider the following integrals:

$$\int_1^{+\infty}\frac{dx}{x} \text{ and } \int_1^{2}\frac{dx}{x}.$$

□ The first integral can be interpreted as an improper integral while the second one is a proper integral. Similarly to a proper integral an improper integral is equal to the area under a curve. But despite the case when the integral of a bounded function over a bounded interval (pic. 4.1) can be taken a plane region whose area is calculated by an improper integral is always infinite (pic. 4.16, *a*).



Pic. 4.16

The problem is to identify whether $\int_1^{+\infty}\frac{dx}{x}$ converges or diverges. Applying the definition, we get

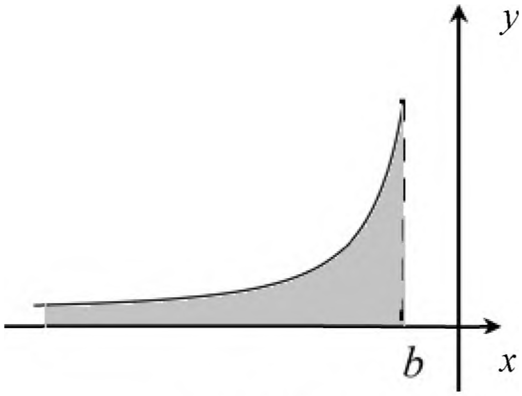
$$\int_1^{+\infty}\frac{dx}{x} = \lim_{\eta\to+\infty}\int_1^{\eta}\frac{dx}{x} = \lim_{\eta\to+\infty}\left(\ln|x|\Big|_1^{\eta}\right) = \lim_{\eta\to+\infty}(\ln\eta) = \infty.$$

Since the limit is infinite the improper integral $\int_1^{+\infty}\frac{dx}{x}$ diverges. Calculations under the limit sign are carried out by means of the Newton–Leibniz formula as the integral transforms into a proper integral. ■

Some variations of the integral $\int_a^{+\infty} f(x)dx$, where $f(x)$ is bounded on the interval $[a,+\infty)$, exist.

1. $\int_{-\infty}^{b} f(x)dx = \lim_{\eta\to-\infty}\int_{\eta}^{b} f(x)dx$,

where $f(x)$ is bounded on the interval $(-\infty,b]$, but the region is unbounded, because the left endpoint is infinite (pic. 4.17).



Pic. 4.17

2. $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$,

where $f(x)$ is bounded on $(-\infty,+\infty)$. In the considered case it can be identified two special points: $-\infty,+\infty$. Dealing with similar cases the given integral should be represented as a sum of two improper integrals with only one special point. Notice, the integral on the left hand side of the equality converges if and only if both integrals on the right hand side converge simultaneously. All of the given integrals are called ***improper integrals of the first kind***.

Example 4.68. Investigate the values of p for which the integral converges. $\int_1^{+\infty}\frac{dx}{x^p}$, where p is a constant, $p>0, p\neq 1$.

□ According to the definition, we have

$$\int_1^{+\infty}\frac{dx}{x^p}=\lim_{\eta\to+\infty}\int_1^{\eta}\frac{dx}{x^p}=\lim_{\eta\to+\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^{\eta}.$$

The problem of evaluating the limit can be separated into two problems depending on values of p.

If $p>1$ $\lim\limits_{\eta\to+\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^{\eta}=\lim\limits_{\eta\to+\infty}\left(\frac{1}{p-1}-\frac{1}{p-1}\frac{1}{\eta^{p-1}}\right)=\frac{1}{p-1}$. Since the limit is a constant the integral converges.

If $0<p<1$ $\lim\limits_{\eta\to+\infty}\left.\frac{x^{1-p}}{1-p}\right|_1^{\eta}=\lim\limits_{\eta\to+\infty}\left(\frac{\eta^{1-p}}{1-p}-\frac{1}{1-p}\right)=\infty$. Hence the integral diverges. ■

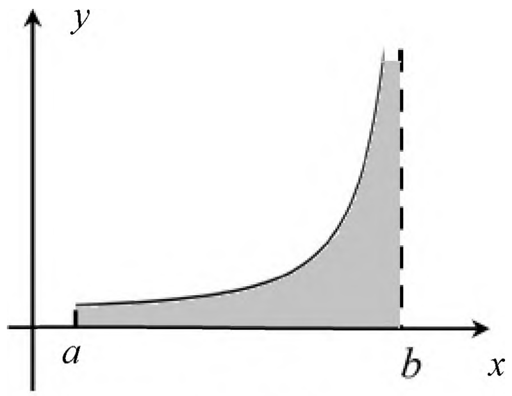
Thus we has proved that

$$\int_1^{+\infty}\frac{dx}{x^p}=\begin{cases}\text{converges, if } p>1,\\ \text{diverges, if } 0<p\le 1\end{cases}$$

Def.: Suppose, $f(x)$ defined on an interval $[a,b)$ is integrable on any closed interval $[a,\eta]\subset[a,b]$. It is assumed that the right endpoint b is a constant.
The quantity

$$\int_a^b f(x)dx=\lim_{\delta\to+0}\int_a^{b-\delta}f(x)dx,$$

if the limit exists, is called an ***improper integral*** of $f(x)$ over the interval $[a,b)$. The essence of this definition is that in any neighborhood of b the function $f(x)$ may happen to be unbounded (pic. 4.18).

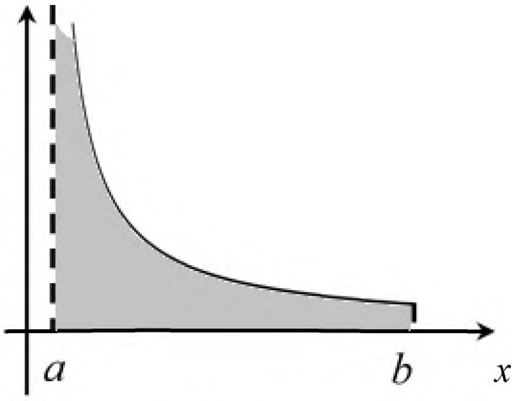


Pic. 4.18

Consider some variations of the improper integral $\int_a^b f(x)dx$:

1. $\int_a^b f(x)dx = \lim_{\delta\to+0} \int_{a+\delta}^b f(x)dx$,

where a is a constant, $f(x)$ may be unbounded (pic. 4.19).



Pic. 4.19

2. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where a, b are constant. In this case, the additive property is used. Both integrals have only one special point, and $f(x)$ may be unbounded in a neighborhood of c, where c is an interior point of the interval $[a,b]$.

All of the considered integrals contained in the second group are called ***improper integrals of the second kind***.

Example 4.69. Investigate values of the parameter p for which the integral $\int_0^1 \frac{dx}{x^p}$ converges.

□ Consider the case when $p=1$. The limits of integration are constant, but $f(x)$ is unbounded near 0 since $\lim_{x\to+0}\frac{1}{x}=+\infty$. So the integral of the second kind is concerned.

$$\int_0^1 \frac{dx}{x} = \lim_{\delta\to+0}\int_\delta^1 \frac{dx}{x} = \lim_{\delta\to+0}\left(\ln(x)\Big|_\delta^1\right) = \lim_{\delta\to+0}\left(-\ln(\delta)\right) = \infty$$

Hence, if $p=1$, $\int_0^1 \frac{dx}{x^p}$ is convergent.

Now let $p>0,\ p\neq 1$.

$$\int_0^1 \frac{dx}{x^p} = \lim_{\delta\to+0}\int_\delta^1 \frac{dx}{x^p} = \lim_{\delta\to+0}\left(\frac{x^{1-p}}{1-p}\Big|_\delta^1\right).$$

If $0 < p < 1$, $\lim\limits_{\delta\to+0}\left(\left.\dfrac{x^{1-p}}{1-p}\right|_{\delta}^{1}\right) = \lim\limits_{\delta\to+0}\left(\dfrac{1}{1-p} - \dfrac{\delta^{1-p}}{1-p}\right) = \dfrac{1}{1-p} < \infty$.

Thus if $0 < p < 1$, $\int\limits_0^1 \dfrac{dx}{x^p}$ is convergent.

If $p > 1$, $\lim\limits_{\delta\to+0}\left(\left.\dfrac{x^{1-p}}{1-p}\right|_{\delta}^{1}\right) = \lim\limits_{\delta\to+0}\left(\left.-\dfrac{1}{(p-1)x^{p-1}}\right|_{\delta}^{1}\right) = \lim\limits_{\delta\to+0}\left(-\dfrac{1}{(p-1)} + -\dfrac{1}{(p-1)\delta^{p-1}}\right) = \infty$.

Thus if $p > 1$, $\int\limits_0^1 \dfrac{dx}{x^p}$ is divergent. ■

So we proved that $\int\limits_1^{+\infty} \dfrac{dx}{x^p} = \begin{cases} \text{diverges, if } p \geq 1, \\ \text{conveges, if } 0 < p < 1. \end{cases}$

Properties of improper integrals

1. Suppose, $f(x)$ and $g(x)$ are functions defined on $[a,\omega)$ and integrable on every closed interval $[a,\eta] \subset [a,\omega)$, where ω may be a constant or infinity. Moreover assume the improper integrals $\int\limits_a^{\omega} f(x)dx$ and $\int\limits_a^{\omega} g(x)dx$ converge. Then, for any real constants λ_1, λ_2 the improper integral $\int\limits_a^{\omega}(\lambda_1 f(x) + \lambda_2 g(x))dx$ converges and

$$\int\limits_a^{\omega}(\lambda_1 f(x) + \lambda_2 g(x))dx = \lambda_1 \int\limits_a^{\omega} f(x)dx + \lambda_2 \int\limits_a^{\omega} g(x)dx .$$

The expression $\lambda_1 f(x) + \lambda_2 g(x)$ is called a linear combination of $f(x)$ and $g(x)$. Thus, the improper integral of a linear combination of $f(x)$ and $g(x)$ is the linear combination of improper integrals of the considered functions.

2. If $\varphi(t)$ is a smooth strictly monotonic function on $\Delta_t = [\alpha, \beta]$ with $\varphi(\alpha) = a$, $\lim\limits_{t\to\beta}\varphi(t) = \omega$ and $\varphi(\Delta_t) \subset \Delta_x$, $\Delta_x = [a,\omega)$ (the range of φ is a subset of), where ω may be a constant or infinity, then

$$\int\limits_a^{\omega} f(x)dx = \int\limits_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)dt .$$

This formula defines the rule for ***integration by substitution*** for improper integrals.

3. If $u(x)$ and $v(x)$ are continuous on $[a,\omega)$ where ω may be a constant or infinity, and $u'(x)$ and $v'(x)$ are piecewise continuous functions on every closed interval $[a,\eta]\subset[a,\omega)$ then

$$\int_a^\omega u\,dv = u\cdot v\Big|_a^\omega - \int_a^\omega v\,du .$$

The formula is referred to as the rule of ***integration by parts*** for improper integrals. Note, the integrals $\int_a^\omega u\,dv$ and $\int_a^\omega v\,du$ are defined in the improper sense. $u\cdot v\big|_a^\omega$ can be interpreted as $u\cdot v\big|_a^\omega = \lim\limits_{\eta\to\omega}\big(u(\eta)v(\eta)-u(a)v(a)\big)$.

Example 4.70. Find the integral $\int_2^{+\infty}\frac{dx}{x\ln x}$ or verify its divergence.

□ Apparently, the integral is an improper integral of the first kind because of boundedness of the integrand function $f(x)=\frac{1}{x\ln x}$ and infinite interval of integration $[2,+\infty)$. Then

$$\int_2^{+\infty}\frac{dx}{x\ln x} = \lim_{\eta\to+\infty}\int_2^{\eta}\frac{dx}{x\ln x}.$$

Note, $d\ln x = \frac{dx}{x}$. Then let $t=\ln x$, we get

$$\lim_{\eta\to+\infty}\int_2^{\eta}\frac{dx}{x\ln x} = \left|\begin{array}{l} t=\ln x,\\ dt=\dfrac{dx}{x}\\ x=2\Rightarrow t=\ln 2\\ x=\eta\Rightarrow t=\ln\eta \end{array}\right| = \lim_{\eta\to+\infty}\int_{\ln 2}^{\ln\eta}\frac{dt}{t} = \lim_{\eta\to+\infty}\ln|t|\Big|_{\ln 2}^{\ln\eta} = +\infty .$$

Hence, $\int_2^{+\infty}\frac{dx}{x\ln x}$ is divergent. ■

Example 4.71. Find the integral $\int_0^{+\infty} xe^{-x}dx$ or verify its divergence

□ The integral $\int_0^{+\infty} xe^{-x}dx$ is an improper integral of the first kind. Consequently, according to the definition

$$\int_0^{+\infty} xe^{-x}dx = \lim_{\eta\to+\infty}\int_0^{\eta} xe^{-x}dx .$$

The integral $\int_0^{\eta} xe^{-x}dx$ is a proper integral (definite integral) that can be taken by means of the method of integration by parts. Then taking x for u and $e^{-x}dx$ for dv we have

$$\lim_{\eta\to+\infty}\int_0^{\eta} xe^{-x}dx = \begin{vmatrix} u = x & du = dx \\ dv = e^{-x}dx & v = \int e^{-x}dx = -e^{-x} \end{vmatrix} = \lim_{\eta\to+\infty}\left(-xe^{-x}\Big|_0^{\eta} + \int_0^{\eta} e^{-x}dx \right) =$$

$$= \lim_{\eta\to+\infty}\left(-xe^{-x}\Big|_0^{\eta} - e^{-x}\Big|_0^{\eta} \right) = \frac{2}{e}.$$

To calculate the limit $\lim_{\eta\to+\infty}\frac{\eta}{e^{-\eta}}$ the L'Hôpital's rule can be used:

$$\lim_{\eta\to+\infty}\frac{\eta}{e^{-\eta}} = \begin{vmatrix} \eta' = 1 \\ \left(e^{-\eta}\right)' = -e^{-\eta} \end{vmatrix} = \lim_{\eta\to+\infty} -\frac{1}{e^{-\eta}} = 0. \blacksquare$$

In most cases it is not necessary to know an exact value of an integral. We just focus on the fact of convergence or divergence of a considered integral.

For this reason, the following theorem can be applied

Theorem (Comparison Theorem). *Suppose,* $f(x)$ *and* $g(x)$ *are defined on* $[a,\omega)$ *and integrable on every closed interval* $[a,\eta]\subset[a,\omega)$.
Then if $0 \le f(x) \le g(x)$ *on* $[a,\omega)$

1. *Convergence of* $\int_a^{\omega} g(x)dx$ *implies convergence of* $\int_a^{\omega} f(x)dx$.

2. *Divergence of* $\int_a^{\omega} f(x)dx$ *implies divergence of* $\int_a^{\omega} g(x)dx$.

Corollary. *Suppose,* $f(x)$ *and* $g(x)$ *are defined on* $[a,\omega)$ *and integrable on every closed interval* $[a,\eta]\subset[a,\omega)$. *Moreover* $0 \le f(x) \le g(x)$ *on* $[a,\omega)$ *and* $g(x)\ne 0\ \forall x\in[a,\omega)$, $\lim_{x\to\omega}\frac{f(x)}{g(x)} = k$ *exists.*

Then

1. *Convergence of* $\int_a^{\omega} g(x)dx$ *provided that* $0 \le k \le \infty$ *implies convergence of* $\int_a^{\omega} f(x)dx$.

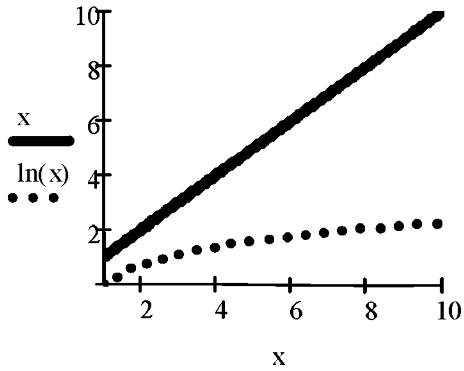
2. *Divergence of* $\int_a^{\omega} g(x)dx$ *provided that* $0 \le k \le \infty$ *implies divergence of* $\int_a^{\omega} f(x)dx$.

In particular case, when $\lim_{x\to\omega}\frac{f(x)}{g(x)}=1$, both integrals $\int_a^{\omega} f(x)dx$ and $\int_a^{\omega} g(x)dx$ converge or diverge simultaneously.

Example 4.72. Examine $\int_2^{+\infty}\frac{dx}{\ln x}$ for convergence.

□ To identify convergence or divergence we should compare behavior of the given integral with an integral whose convergence or divergence is already proved.

Note, the inequality $x > \ln x$ holds for $\forall x \in [2,+\infty)$. This fact can be illustrated graphically (pic. 4.20).



Pic. 4.20

Consequently $\frac{1}{x} < \frac{1}{\ln x}$. Now if let $f(x)=\frac{1}{x}$ and $g(x)=\frac{1}{\ln x}$ then according to the theorem divergence of $\int_2^{+\infty}\frac{dx}{x}$ implies divergence of the given integral $\int_2^{+\infty}\frac{dx}{\ln x}$. ■

Example 4.73. Examine $\int_2^{+\infty}\frac{dx}{\sqrt{x}-\sqrt[3]{x}}$ for convergence.

□ $\frac{1}{\sqrt{x}-\sqrt[3]{x}}$ can be expressed as $\frac{1}{Q(\sqrt{x},\sqrt[3]{x})}$, where $Q(\cdot,\cdot)$ is a polynomial of the listed arguments. As we know the leading term $a_l x^l$ of a polynomial

$P_l(x)=a_lx^l+a_{l-1}x^{l-1}+...+a_0$ determines its behavior at infinity. Thus behavior of $\frac{1}{\sqrt{x}-\sqrt[3]{x}}$ can be compared with behavior of $\frac{1}{\sqrt{x}}$. Indeed

$$\lim_{x\to+\infty}\frac{\frac{1}{\sqrt{x}-\sqrt[3]{x}}}{\frac{1}{\sqrt{x}}}=\lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x}-\sqrt[3]{x}}=1.$$

Since $\int_2^{+\infty}\frac{dx}{\sqrt{x}}$ diverges $\int_2^{+\infty}\frac{dx}{\sqrt{x}-\sqrt[3]{x}}$ also diverges. ■

Absolute and conditional convergence of improper integrals

Def.: The improper integral $\int_a^{\omega} f(x)dx$ ***converges absolutely*** if the integral $\int_a^{\omega}|f(x)|dx$ converges.

Example 4.74. Examine $\int_1^{\infty}\frac{\cos x}{x^2}dx$ for absolute convergence.

□ Consider the integral $\int_1^{\infty}\left|\frac{\cos x}{x^2}\right|dx$. $\left|\frac{\cos x}{x^2}\right|$ takes only nonnegative values. So we can apply the Comparison theorem. Since $|\cos x|\le 1$, $\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$. Then $\int_1^{\infty}\left|\frac{\cos x}{x^2}\right|dx$ converges because $\int_1^{\infty}\frac{dx}{x^2}$ converges. Hence the given integral $\int_1^{\infty}\frac{\cos x}{x^2}dx$ converges absolutely. ■

Def.: If an improper integral converges but not absolutely, we say that it ***converges conditionally***.

Consider the integral $\int_{\frac{\pi}{2}}^{+\infty}\frac{\sin x}{x}dx$. The integral is an improper integral of the first kind. Applying the definition and the method of integration by parts for improper integrals we have

$$\int_{\frac{\pi}{2}}^{+\infty}\frac{\sin x}{x}dx=\lim_{\eta\to\infty}\int_{\frac{\pi}{2}}^{\eta}\frac{\sin x}{x}dx=\left|\begin{matrix}u=\frac{1}{x} & du=-\frac{dx}{x^2}\\ dv=\sin x dx & v=-\cos x\end{matrix}\right|=$$

$$=\lim_{\eta\to\infty}\left(-\frac{\cos x}{x}\Bigg|_{\frac{\pi}{2}}^{\eta}-\int_{\frac{\pi}{2}}^{\eta}\frac{\cos x}{x^2}dx\right)=\lim_{\eta\to\infty}\left(-\frac{\cos\eta}{\eta}-0\right)-\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos x}{x^2}dx,$$

$\lim_{\eta\to\infty}\left(-\frac{\cos\eta}{\eta}\right)=\lim_{\eta\to\infty}\left(-\cos\eta\cdot\frac{1}{\eta}\right)=\left|\begin{matrix}\cos\eta \text{ is bounded}\\ \frac{1}{\eta}\to 0, \eta\to\infty\end{matrix}\right|=0$. Absolute convergence of $\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos x}{x^2}dx$ was proved earlier.

However, the integral $\int_{\frac{\pi}{2}}^{+\infty}\frac{\sin x}{x}dx$ doesn't converge absolutely. Indeed,

$$\int_{\frac{\pi}{2}}^{+\infty}\left|\frac{\sin x}{x}\right|dx\geq\int_{\frac{\pi}{2}}^{+\infty}\frac{\sin^2 x}{x}dx=\int_{\frac{\pi}{2}}^{+\infty}\frac{1-\cos 2x}{2x}dx=\frac{1}{2}\int_{\frac{\pi}{2}}^{+\infty}\frac{dx}{x}-\frac{1}{2}\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos 2x}{x}dx.$$

The integral $\int_{\frac{\pi}{2}}^{+\infty}\frac{dx}{x}$ diverges and the integral $\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos 2x}{x}dx$ diverges. So if at least one of the integrals on the right hand side of the inequality diverges then the integral on the left hand side diverges as well.

Convergence of $\int_{\frac{\pi}{2}}^{+\infty}\frac{\cos 2x}{x}dx$ can be verified by use of Dirichlet's test. Dirichlet's test claims that if there exists a number M such that $\left|\int_{a}^{\eta}f(x)dx\right|\leq M$ for every $\eta\in[a,+\infty)$ and $g(x)$ is monotonically decreasing provided that $\lim_{x\to+\infty}g(x)=0$ then $\int_{a}^{+\infty}f(x)g(x)dx$ converges.

Exercises

1. Evaluate the integrals or verify its divergence:

a) $\int_{-\infty}^{+\infty} e^x dx$;

b) $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$;

c) $\int_{0}^{2} \frac{dx}{(x-2)^2}$;

d) $\int_{0}^{+\infty} x \sin x dx$.

2. Examine the integrals for convergence:

a) $\int_{2}^{\infty} \frac{dx}{x \ln^3 x}$;

b) $\int_{0}^{\infty} \frac{dx}{x^2+2x+2}$;

c) $\int_{1}^{2} \frac{dx}{\sqrt[3]{x-1}}$;

d) $\int_{1}^{e} \frac{dx}{x \ln x}$;

e) $\int_{0}^{\infty} \frac{x^3 dx}{2+x^4}$;

f) $\int_{0}^{1} \frac{dx}{\sqrt[3]{(1-x)^2}}$.

REFERENCES

1. Demidovich B.P. Sbornik zadach i uprazhnenij po matematicheskomu analizu. – M.: Nauka, 2002.
2. Il'in V.A., Poznyak E.G. Osnovy matematicheskogo analiza: v 2 ch. – M.: Nauka, 1971.
3. Kremer N.Sh., Putko B.A., Trishin I.M., Fridman M.N. Vysshaya matematika dlya ekonomicheskih special'nostej: uchebnik i praktikum (chast' I i II)/ pod red. prof. N.Sh. Kremera. – 2-e izd.,pererab. i dop. – M.: Vysshee obrazovanie, 2008.
4. Kudryavcev L.D. Kurs matematicheskogo analiza. v 3-h t. – M.: Drofa, 2004.
5. Nikol'skij S.M. Kurs matematicheskogo analiza. – M.: Fizmatlit, 2001.
6. Savost'yanova N.I., Fedorova N.M. Matematicheskij analiz dlya inzhenerov i ekonomistov/ pod red. prof. A.V. Panteleeva.– M.: Faktorial, 2018.
7. Fihtengol'c G.M. Kurs differencial'nogo i integral'nogo ischisleniya. – M.: Fizmatlit, 2005.
8. Zorich V. Mathematical Analysis I. – Springer, 1980.
9. Piskunov N. Differential and Integral Calculus. – M.: Mir Publisher, 1969.
10. Thomas G.B.,Weir M. D, Hass J. Thomas' Calculus: Early Transcendentals. – Pearson, 2014.
11. Stewart J. Calculus Early Transcendentals. – Thomson, 2008.
12. Apostol Tom M.. Calculus, Volume 1: One-Variable Calculus with an Introduction to Linear Algebra. – John Wiley & Sons, 1967.

БИБЛИОГРАФИЧЕСКИЙ СПИСОК

1. *Демидович Б.П.* Сборник задач и упражнений по математическому анализу. – М.: Наука, 2002.
2. *Ильин В.А., Позняк Э.Г.* Основы математического анализа: в 2 ч. – М.: Наука, 1971.
3. *Кремер Н.Ш., Путко Б.А., Тришин И.М., Фридман М.Н.* Высшая математика для экономических специальностей: учебник и практикум (часть I и II)/ под ред. проф. Н.Ш. Кремера. – 2-е изд., перераб. и доп. – М.: Высшее образование, 2008.
4. *Кудрявцев Л.Д.* Курс математического анализа. В 3-х т. – М.: Дрофа, 2004.
5. *Никольский С.М.* Курс математического анализа. – М.: Физматлит, 2001.
6. *Савостьянова Н.И., Федорова Н.М.* Математический анализ для инженеров и экономистов / под ред. проф. А.В. Пантелеева. – М.: Факториал, 2018.
7. *Фихтенгольц Г.М.* Курс дифференциального и интегрального исчисления. – М.: Физматлит, 2005.
8. Zorich V. Mathematical Analysis I. – Springer, 1980.
9. Piskunov N. Differential and Integral Calculus. – M.: Mir Publisher, 1969.
10. Thomas G.B.,Weir M. D, Hass J. Thomas’ Calculus: Early Transcendentals. – Pearson, 2014.
11. Stewart J. Calculus Early Transcendentals. – Thomson, 2008.
12. Apostol Tom M.. Calculus, Volume 1: One-Variable Calculus with an Introduction to Linear Algebra. – John Wiley & Sons, 1967.

Учебное издание

Кудрявцева Ирина Анатольевна

МАТЕМАТИЧЕСКИЙ АНАЛИЗ: ДИФФЕРЕНЦИАЛЬНОЕ И ИНТЕГРАЛЬНОЕ ИСЧИСЛЕНИЕ ФУНКЦИИ ОДНОЙ ПЕРЕМЕННОЙ

УЧЕБНОЕ ПОСОБИЕ

Издательство «Доброе слово»

Заказ книг: http://www.dobroeslovo.info

Подписано в печать:

П.л. 10. Формат 60x90/16

Тираж 100 экз.